DYNAMIC RESPONSE OF A WETTING LIQUID ENCLOSED IN A ROTATING TANK IN A ZERO-G ENVIRONMENT AND SUBJECTED TO VARIOUS DISTURBING FORCES

By

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TO MY LOVE

JULIET

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PARTIAL LIST OF SYMBOLS

t	time
Ω	constant angular speed of the rotating tank
$2L$	length of the tank
R_o	radius of the cylinder
C	semi-minor-axis of the undisturbed bubble
$2\ell=2L/C$	dimensionless length of the tank
$\eta_o=R_o/C$	dimensionless radius of the cylinder
(r,θ,z)	cylindrical coordinates
(η,θ,ζ)	dimensionless cylindrical coordinates, $\eta = r/C$, $\zeta = z/C$
(α,β,θ)	dimensionless spheroidal coordinates
$(\hat{u},\hat{v},\hat{w})$	perturbation velocity components with respect to the (r,θ,z) directions
(u,v,w)	dimensionless perturbation velocity components, corresponding to the (η,θ,ζ) directions
$\hat{p}$	pressure
p	dimensionless perturbation pressure
ρ	density of the liquid
T	coefficient of the surface tension
J	curvature of the undisturbed bubble
J^*	curvature of the perturbed bubble
α_1	perturbation amplitude of the equilibrium bubble interface
ω	circular frequency of oscillation of the interface
ϵ	magnitude of acceleration in a constant small transverse force field
K	ratio of the semi-major axis to the semi-minor axis of the original, undisturbed bubble
i	imaginary unit ($i^2 = -1$)

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Major Department: Engineering Science and Mechanics

During the past few years, a growing interest in the investigation of various problems in the area of the mechanics of contained fluids under reduced effective gravity has been observed, stimulated by the practical needs of spacecraft operations. Among others, the question of the dynamic response and stability of a large body of fluid in a tank placed into subgravity, is one of considerable importance. The matter of concern has to do with the storage, transportation, and utilization of liquid propellants in the operation of spacecraft powerplants.

The present study deals with the exploration of the nature of liquid interface oscillations in a rotating tank and the possibility of rotational stabilization.

The system under consideration consists of a right, circular, cylindrical tank of finite length, partly filled with an inviscid, incompressible, wetting liquid. The whole system spins about the tank axis with constant angular speed in a near zero-g environment.

The large scale vapor bubble takes a stable equilibrium shape which is similar to an elongated spheroid and located symmetrically about the axis of rotation. When perturbing forces are prevailing, the liquid body and the free interface will oscillate about the equilibrium configuration.

The equations of motion are linearized. Surface tension and rotational velocity component are both essential in the treatment of the problem and are taken fully into account.

The self-sustained oscillations are governed by an elliptic differential equation for the perturbation pressure field. In this case, the bubble is approximated by a prolate spheroid which is embedded into a spheroidal coordinate system. Hence associated Legendre functions can be employed for the series expansion of the solution. A special method has been employed in order to account for the homogeneous boundary condition at the walls of the tank. The resulting eigenvalue problem for the relative frequency of oscillation is nonlinear. The eigenfrequencies are obtained through iteration procedure. They are all real and greater than two. The first two eigenfrequencies can be computed with great accuracy. The oscillations are stable. The amplitude of oscillation is large near the equator of the bubble. For very slowly rotating systems the eigenfrequencies tend to accumulate near the critical frequency.

For a forced oscillation, induced by a reduced gravity field of constant magnitude and direction in an inertial frame of reference, the relative frequency of oscillation is less than two. The governing differential equation is hyperbolic, hence the method of analysis is

different. Consequently the structure of the flow field is different from that of the above mentioned elliptic case. The mathematical problem is transformed into integro-differential equations which can be integrated numerically by means of Picard's method of successive approximations. It is found that a steady solution in a cylindrical tank of finite length does not exist. However, a solution exists in a cylindrical tank of infinite length. The perturbations extend to infinity. A numerical example for this forced oscillation problem has been carried out.

We find from this analysis that the rotation of the system has a profound influence on the dynamic response of the liquid body. For a system without rotation, a small disturbance is liable to cause instability of the equilibrium configuration. The surface tension provides only limited means to damp out any dynamic disturbance. For a system with rotation, all the small oscillations of the interface are stable. A small disturbance will be readily absorbed dynamically without destroying the stable configuration. If viscous effects were taken into account, the oscillations can be effectively dissipated through the induced secondary flow.

CHAPTER I

INTRODUCTION

During the past few years, a growing interest in the investigation of various problems in the area of the mechanics of contained fluids under reduced effective gravity has been observed, stimulated by the practical needs of spacecraft operations. Among others, the question of the dynamic response and stability of a large body of fluid in a tank placed into subgravity, is one of considerable importance. The matter of concern has to do with the storage, transportation, and utilization of liquid propellants in the operation of spacecraft powerplants.

In low gravitational environments, forces like surface tension and the centrifugal force, induced by slow rotation of the fluid, will have a dominating effect on the large scale equilibrium configuration and the dynamics of a fluid system. For example, a right circular cylindrical tank, partially filled with a wetting liquid, spins about its axis of revolution with constant angular speed and is placed in a weak gravitational (zero-g) field. Then the surface tension at the interface, together with the centrifugal force, causes the vapor cavity (bubble) inside the vessel to take an elongated spheroid-like shape, situated symmetrically about the axis of rotation. Equilibrium configurations for various constant angular speeds have been studied by

Rosenthal [1].[*] A concise survey on the literature to this field is given in a review article by Habip [2].

The present study deals with the exploration of the nature of liquid interface oscillations in a rotating tank and the possibility of rotational stabilization.

The chosen frame of reference is fixed in the tank and rotates with it. The perturbation velocities, the interface wave amplitude, and the disturbing forces are assumed to be small. Thus, in the equations of motion, the equation of continuity, and the boundary conditions terms involving quadratic or higher orders of the perturbation quantities are neglected.

The fluid is assumed to be inviscid and incompressible. Surface tension and rotational velocity components are both essential in the study. They are taken fully into account as far as a linearized theory will allow.

For this rotating fluid oscillation problem there are two frequency ranges for which the disturbances are of an entirely different character. Let ω be the dimensionless frequency of oscillation which is measured relative to the constant rotation of the tank-liquid system. Then for $\omega > 2$, the type of the governing partial differential equation is elliptic, while for $\omega < 2$, the flow field is described by a hyperbolic equation. The method of analysis and the physical interpretation of the flow phenomena are entirely different.

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[*] Numbers in brackets refer to the Bibliography.

For the elliptic case, the bubble is approximated by a prolate spheroid which is embedded into a spheroidal coordinate system. Hence, associated Legendre functions can be employed for the series expansion of the solution. A special method has been employed in order to account for the homogeneous boundary condition at the wall of the tank. The resulting eigenvalue problem for the relative frequency of oscillation is nonlinear, since the eigenfrequency is included as a parameter in the formulation of the differential equation as well as the boundary conditions. The eigenfrequencies are obtained through a successive approximation procedure. This procedure proved to be stable for the numerical computations performed for a sequence of examples.

The result for this case indicates, that for the circumferential mode $m = 2$, the first eigenfrequency greater than two corresponds to the first mode of vibration in the meridian plane. Hence, for higher modes of oscillation the flow field is certainly elliptic in nature.

For a forced oscillation, induced by a reduced gravity field of constant magnitude and direction in an inertial frame of reference, the relative frequency of oscillation is less than two. The governing differential equation is hyperbolic, hence the method of analysis is different. Consequently, the structure of the flow field is different from that of the above mentioned elliptic case. The mathematical problem (Cauchy and Goursat problem) is transformed into integro-differential equations which can be integrated numerically by means of Picard's method of successive approximations. It is found that a steady solution in a cylindrical tank of finite length does not exist. However, a solution exists in a cylindrical tank of infinite length. The perturbations extend to infinity. A numerical example for this forced oscillation problem has been carried out.

In the present work we consider only oscillations which are symmetric with respect to the equatorial plane of the bubble, i.e., symmetric with respect to the variable ζ in the cylindrical coordinate system, and with respect to the variable β in the prolate spheroidal coordinate system.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Derivation of the linearized equations of motion

The equations of motion governing the fluid flow in a rotating frame of reference can be written in the following form [3]

$$\frac{\partial \vec{q}}{\partial t} + 2\vec{\Omega}\times\vec{q} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + \vec{q}\cdot\nabla\vec{q} = \vec{F} - \frac{1}{\rho}\nabla\hat{p} \ , \tag{1}$$

where $\vec{q}$ denotes the relative (perturbation) velocity vector, $\vec{\Omega}$ the angular velocity vector, $\vec{r}$ the position vector from the origin of the rotating coordinate system to the point occupied by the fluid particle, and $\vec{F}$ the external force per unit mass of the fluid. We further denote the density of the liquid by ρ and the pressure by $\hat{p}$. In the present problem the basic fluid flow is in a state of rigid body rotation with respect to the inertia frame of reference and hence is at rest with respect to the rotating, nontranslating frame as referred to in this investigation. By neglecting the quadratic terms of the perturbation velocity components a linearized system of equations is obtained. This system can be written in a cylindrical coordinate system (r,θ,z) as

$$\left\{\begin{aligned} &\frac{\partial \hat{u}}{\partial t} - 2\Omega\hat{v} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial r} + F_r \ , && \text{(1a)}\\ &\frac{\partial \hat{v}}{\partial t} + 2\Omega\hat{u} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \theta} + F_\theta \ , && \text{(1b)}\\ &\frac{\partial \hat{w}}{\partial t} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial z} + F_z \ , && \text{(1c)}\end{aligned}\right.$$

with $\{\hat{u},\hat{v},\hat{w}\}$ as the components of the perturbation velocity vector and with $\{F_r,F_\theta,F_z\}$ as the components of the external force.

The equation of continuity reads

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{u}) + \frac{1}{r}\frac{\partial \hat{v}}{\partial \theta} + \frac{\partial \hat{w}}{\partial z} = 0 \quad . \tag{2}$$

The system of differential equations can be made dimensionless by choosing the semi-minor axis of the undisturbed bubble radius C as the length scale and Ω^{-1} as the time scale. Further, the acting external perturbing forces in our case are assumed to originate from an oscillatory force field which is transverse to the direction of the tank axis. It is assumed further that this field has a constant direction in the inertia frame of reference. Then the pressure can be written as

$$\hat{p} = \frac{1}{2}\rho\Omega^2 c^2 \{\eta^2 + 2p\} + \epsilon\rho c\eta e^{i(\Omega t+\theta)} + \hat{p}_{oL} , \tag{3}$$

where $\hat{p}_{oL}$ is a constant and the time dependent, second term of the right-hand side takes account of the perturbation force effects. The equations of motion are then reduced to

$$\left\{\begin{aligned} \frac{\partial u}{\partial t} - 2v &= -\frac{\partial p}{\partial \eta} , && \text{(4a)} \\ \frac{\partial v}{\partial t} + 2u &= -\frac{1}{\eta}\frac{\partial p}{\partial \theta} , && \text{(4b)} \\ \frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial \zeta} , && \text{(4c)} \end{aligned}\right.$$

while the equation of continuity becomes

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta u) + \frac{1}{\eta}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial\zeta} = 0. \tag{5}$$

2.2 Derivation of the field equation

By means of simple elimination, relations between each velocity component and the pressure may be obtained from the equations of motion, namely,

$$\left\{\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} + 4\right) u = -\frac{\partial^2 p}{\partial t\partial\eta} - \frac{2}{\eta}\frac{\partial p}{\partial\theta} , && \text{(6a)}\\ &\left(\frac{\partial^2}{\partial t^2} + 4\right) v = -\frac{1}{\eta}\frac{\partial^2 p}{\partial t\partial\eta} + 2\frac{\partial p}{\partial\eta} , && \text{(6b)}\\ &\qquad\frac{\partial^2 w}{\partial t^2} = -\frac{\partial^2 p}{\partial t\partial\zeta} . && \text{(6c)}\end{aligned}\right.$$

Now these relations can be used to eliminate the velocity components from the equation of continuity (5). Thus, the governing field equation for the fluid flow is

$$\frac{\partial^2}{\partial t^2}\left\{\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial p}{\partial\eta}\right) + \frac{1}{\eta^2}\frac{\partial^2 p}{\partial\theta^2} + \frac{\partial^2 p}{\partial\zeta^2}\right\} + 4\frac{\partial^2 p}{\partial\zeta^2} = 0 . \tag{7}$$

This partial differential equation exhibits some very distinct properties. Let us assume that the perturbation pressure can be written in the form

$$p(\eta,\theta,\zeta,t) = P(\eta,\theta,\zeta)e^{i\omega t} , \tag{8}$$

where, of course, the real part of the right-hand side has to be used for physical interpretation.

With this setup the field equation for the perturbation pressure is transformed into

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial p}{\partial\eta}\right)+\frac{1}{\eta^2}\frac{\partial^2 p}{\partial\theta^2}+\frac{\omega^2-4}{\omega^2}\frac{\partial^2 p}{\partial\zeta^2}=0 , \tag{9}$$

and it follows that the foregoing equation is of elliptic type, when $\omega > 2$, and of hyperbolic type, when $\omega < 2$. The last term on the left-hand side of (9) vanishes if $\omega = 2$. Similarly, the velocity components can then be expressed explicitly in terms of the perturbation pressure and its derivatives:

$$\begin{cases} (\omega^2-4)\,u = \left(i\omega\dfrac{\partial p}{\partial\eta}+\dfrac{1}{\eta}\dfrac{\partial p}{\partial\theta}\right)e^{i\omega t} , & \text{(10a)} \\[2ex] (\omega^2-4)\,v = \left(i\omega\dfrac{1}{\eta}\dfrac{\partial p}{\partial\theta}-2\dfrac{\partial p}{\partial\eta}\right)e^{i\omega t} , & \text{(10b)} \\[2ex] \omega^2 w = i\omega\dfrac{\partial p}{\partial\zeta}\,e^{i\omega t} . & \text{(10c)} \end{cases}$$

Now let the perturbation velocity (u,v,w) be divided into two parts (u_1, v_1, w_1) and (u_2, v_2, w_2), such that

$$u_1 = i\omega\frac{\partial p}{\partial\eta}e^{i\omega t} , \quad v_1 = i\omega\frac{1}{\eta}\frac{\partial p}{\partial\theta}e^{i\omega t} , \quad w_1 = i\omega\frac{\partial p}{\partial\zeta}e^{i\omega t} ,$$

$$u_2 = \left(\frac{2}{\omega^2-4}\frac{1}{\eta}\frac{\partial p}{\partial\theta}+\frac{4i}{\omega(\omega^2-4)}\frac{\partial p}{\partial\eta}\right)e^{i\omega t} , \quad v_2 = \left(\frac{-2}{\omega^2-4}\frac{\partial p}{\partial\eta}+\frac{4i}{\omega(\omega^2-4)}\frac{1}{\eta}\frac{\partial p}{\partial\theta}\right)e^{i\omega t} ; \quad w_2 = 0 .$$

The velocity component (u_1,v_1,w_1) is irrotational because it is the gradient of a scalar function. For the velocity component (u_2,v_2,w_2), we have

$$\text{curl}\,\vec{v} = \text{curl}\,\vec{v}_2 =$$

$$= \left(\left[\frac{-4i}{\omega(\omega^2-4)} \frac{1}{\eta} \frac{\partial^2 p}{\partial\theta\partial\zeta} + \frac{2}{\omega^2-4} \frac{\partial^2 p}{\partial\eta\partial\zeta} \right] e^{i\omega t}, \left[\frac{4i}{\omega(\omega^2-4)} \frac{\partial^2 p}{\partial\eta\partial\zeta} + \frac{2}{\omega^2-4} \frac{1}{\eta} \frac{\partial^2 p}{\partial\theta\partial\zeta} \right] e^{i\omega t}, \left[\frac{2}{\omega^2} \frac{\partial^2 p}{\partial\zeta^2} \right] e^{i\omega t} \right),$$

where

$$\vec{V} = (u, v, w) \quad \text{and} \quad \vec{V}_2 = (u_2, v_2, w_2) \, .$$

Hence, the velocity field can be characterized as follows:

a) For $\omega > 2$, the magnitude of the rotational velocity component is smaller than the magnitude of the irrotational one. Their ratio tends to zero at the rate $1/\omega$ as ω tends to infinity.

b) For $\omega < 2$, the magnitude of the rotational velocity component is larger than the magnitude of the irrotational one. Their ratio tends to infinity of the order $O(1/\omega^2)$ as ω tends to zero. The vorticity effect is so dominating that the velocity field may have to take up a cellular structure in order to make the flow in the fluid domain dynamically possible. The hyperbolic type of the field equation permits such discontinuities in its solution. An example can be found in Phillips [4].

c) For $\omega = 2$, the partial differential equation is indeterminate in the sense that the ζ-dependence is arbitrary. Any solution of the form

$$p(\eta, \theta, \zeta, t) = \left\{ K(\eta, \theta)\, \varpi_1(\zeta) + \varpi_2(\zeta) \right\} e^{i\omega t} ,$$

where $K(\eta, \theta)$ satisfies the field equation and the boundary conditions and any functions $\varpi_1(\zeta)$ and $\varpi_2(\zeta)$ which fulfill the boundary conditions are admissible. The solution is not unique. Hence, in the linearized theory such a flow is unstable.

For the case $\omega > 2$ the affine transformation of the space coordinates (Figure 1)

$$\begin{cases} \eta = \eta \,, \\ \theta = \theta \,, \\ \mu = \dfrac{\omega}{\sqrt{\omega^2-4}} \, \zeta \,, \end{cases} \tag{11}$$

will reduce the field equation (9) to the Laplace equation

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \rho}{\partial \eta}\right) + \frac{1}{\eta^2}\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \rho}{\partial \mu^2} = 0. \tag{12}$$

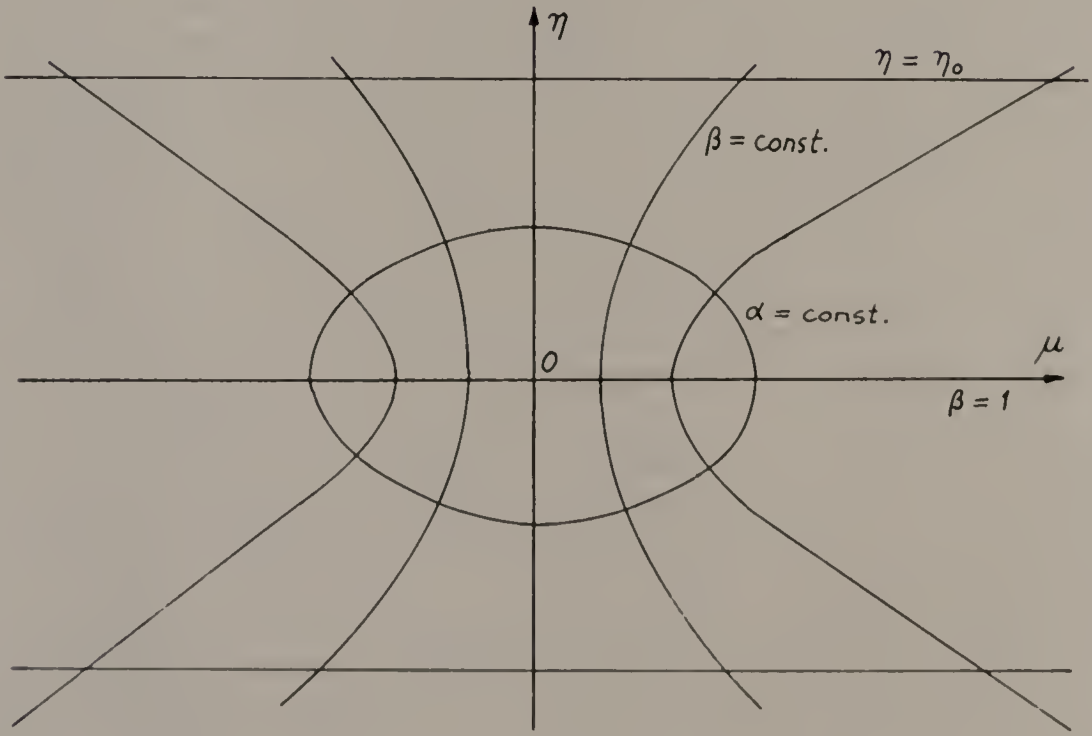

Figure 1. A meridian section of the deformed cylindrical coordinate system (η , θ , μ) and the prolate spheroidal coordinate system (α , β , θ).

Another coordinate transformation is needed for the analysis of this problem. The cylindrical coordinates (η, θ, μ) are transformed into the prolate spheroidal coordinates (α, β, θ) by means of

$$\left\{\begin{array}{ll} \eta = \frac{a}{2}\sqrt{(\alpha^2-1)(1-\beta^2)} & , \\ \mu = \frac{a}{2}\alpha\beta & , \\ \theta = \theta & , \end{array}\right. \tag{13}$$

where a is a scale constant which will be determined later.

In this coordinate system the Laplace equation for the pressure field becomes

$$\frac{1}{\alpha^2-\beta^2}\left\{\frac{\partial}{\partial\alpha}\left[(\alpha^2-1)\frac{\partial p}{\partial\alpha}\right] + \frac{\partial}{\partial\beta}\left[(1-\beta^2)\frac{\partial p}{\partial\beta}\right] + \left[\frac{1}{\alpha^2-1} - \frac{1}{1-\beta^2}\right]\frac{\partial^2 p}{\partial\theta^2}\right\} = 0. \tag{14}$$

It should be noted that the field equation is separable in the cylindrical as well as in the prolate spheroidal coordinate system.

2.3 The boundary conditions for the elliptic case

The next step in our analysis is to formulate the boundary conditions at the rigid tank wall and the free liquid vapor interface.

At the plane end disks of the tank the normal velocity, which is in the μ-direction, must vanish. Thus

$$\frac{\partial p}{\partial\mu} = 0 \qquad \text{at} \qquad \mu = \frac{\pm\,\omega\ell}{\sqrt{\omega^2-4}} \;. \tag{15}$$

At the cylindrical portion of the tank wall the normal velocity component, which is in the η-direction, must vanish. Hence,

$$i\omega\frac{\partial p}{\partial\eta} + \frac{2}{\eta}\frac{\partial p}{\partial\theta} = 0 \qquad at \qquad \eta = \eta_0 \;. \tag{16}$$

The bubble interface shall first be specified in the spheroidal coordinate system before the boundary conditions are set up. The approximate bubble shape is assumed to be a spheroid which has the same ratio of the semimajor axis to the semiminor axis as the exact shape of the bubble. The semiminor axis is normalized to one in the dimensionless coordinate systems. By taking into account the stretching due to the affine coordinate transformation the bubble can be embedded into the spheroidal coordinate system as one of the coordinate surfaces $\alpha = \alpha_o$ = constant, where α_o is given by

$$\frac{\alpha_o}{\sqrt{\alpha_o^2 - 1}} = K \frac{\omega}{\sqrt{\omega^2 - 4}} \quad . \tag{17}$$

The scale constant a which appears in the relations defining the coordinate transformation between (η, θ, μ) and (α, β, ρ) is then

$$a = \frac{2}{\sqrt{\alpha_o^2 - 1}} \quad . \tag{18}$$

Hence the perturbation interface can be written in the form

$$\alpha = \alpha_o + \alpha_1 (\beta, \theta) e^{i\omega t} \quad , \tag{19}$$

where the function $\alpha_1(\beta, \theta)$ is small such that $\| \alpha_1 \| << (\alpha_o^2 - 1)$, where $\| \alpha_1 \|$ denotes the norm of α_1.

Now a bounding surface $F(\alpha, \beta, \theta, t) = 0$ of a continuous medium is a material surface. Mathematically this condition can be written as

$$\frac{d}{dt} \left\{ F(\alpha.\beta, \theta, t) \right\} = 0 \quad , \tag{20}$$

with

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \alpha} \frac{d\alpha}{dt} + \frac{\partial F}{\partial \beta} \frac{d\beta}{dt} + \frac{\partial F}{\partial \theta} \frac{d\theta}{dt} \quad . \tag{21}$$

In the present case we have

$$F(\alpha,\beta,\theta,t) \equiv \alpha - \alpha_0 - \alpha_1(\beta,\theta)e^{i\omega t} = 0 \quad , \tag{22}$$

hence

$$\frac{dF}{dt} = \frac{d\alpha}{dt} - i\omega\alpha_1(\beta,\theta)e^{i\omega t} - \frac{\partial\alpha_1}{\partial\theta}\frac{d\theta}{dt}e^{i\omega t} - \frac{\partial\alpha_1}{\partial\beta}\frac{d\beta}{dt}e^{i\omega t} = 0. \tag{23}$$

Neglecting quadratic terms in the above equation there results

$$\frac{d\alpha}{dt} - i\omega\alpha_1(\beta,\theta)\,e^{i\omega t} = 0 \; . \tag{24}$$

The perturbation velocity component in the α-direction, v_α, may be written in terms of the pressure and its derivatives. It turns out that

$$v_\alpha = h_\alpha \frac{d\alpha_1}{dt} \;:\; h_\alpha = \frac{a}{2}\,\frac{\alpha^2-\beta^2}{\alpha^2-1}; \quad (\omega^2-4)\frac{d\alpha_1}{dt} = \frac{4i\omega}{a^2}\,\frac{(\alpha^2-1)}{(\alpha^2-\beta^2)}\frac{\partial \ell}{\partial\alpha} + \frac{8\alpha}{a^2(\alpha^2-\beta^2)}\frac{\partial \ell}{\partial\theta}, \tag{25}$$

where h_α denotes a scale factor. Then the above results can be summarized in the following equation:

$$(\omega^2-4)\alpha_1(\beta,\theta)e^{i\omega t} = \frac{4(\alpha^2-1)}{a^2(\alpha^2-\beta^2)}\frac{\partial \ell}{\partial\alpha} - \frac{i}{\omega}\,\frac{4\,(2\alpha)}{a^2(\alpha^2-\beta^2)}\,\frac{\partial \ell}{\partial\theta} \; . \tag{26}$$

A dynamical boundary condition shall also be satisfied besides the above kinematical boundary condition. The pressure at both sides of the interface, together with the surface tension force, must be in equilibrium with the centrifugal and inertia forces. Now, the pressure in the liquid has been defined previously by equation (3)

$$\hat{p} = \frac{1}{2}\rho\Omega^2c^2(\eta^2+2p) + \epsilon\rho c\eta\, e^{i(\Omega t+\theta)} + \hat{p}_{oL} \; .$$

In the cavity, the pressure is a constant, $\hat{p}_{oG}$. Then the boundary condition at the interface reads

$$\frac{1}{2}\rho\Omega^2c^2(\eta^2+2p)+\epsilon\rho c\eta e^{i(\theta+\Omega t)}\Big|_{\alpha=\alpha_0+\alpha_1}+(\hat{p}_{oL}-\hat{p}_{oG})=-\frac{1}{c}TJ^*. \tag{27}$$

For the perturbation configuration of the bubble we have

$$\eta=\frac{1}{2}a\left(1+\frac{\alpha_0\alpha_1}{\alpha_0^2-1}\right)\sqrt{(\alpha_0^2-1)(1-\beta^2)}\;, \tag{28}$$

hence

$$\begin{aligned}\eta^2&=\frac{a^2}{4}\left\{(\alpha_0^2-1)(1-\beta^2)\right\}\left(1+\frac{2\alpha_0\alpha_1}{\alpha_0^2-1}\right)\\&=\frac{a^2}{4}(\alpha_0^2-1)(1-\beta^2)+\frac{a^2}{4}\cdot2\alpha_0\alpha_1(1-\beta^2)\;.\end{aligned} \tag{29}$$

The following expression is arrived at by substituting the above quantity into the definition of the pressure in the liquid and by subtracting the quantities which are originally in equilibrium (static equilibrium configuration of the rotating tank-fluid system with no disturbance forces acting) from both sides:

$$\frac{1}{2}\rho\Omega^2c^2\left\{\frac{a^2}{4}\cdot2\alpha_0\alpha_1(1-\beta^2)+2p_0\right\}+\epsilon\rho c\eta e^{i(\Omega t+\theta)}=-\frac{1}{c}T(J^*-J)\cdot \tag{30}$$

We further define the following dimensionless numbers

$$E=\frac{\rho\Omega^2c^3}{8T}\;;\qquad B=\frac{\epsilon\rho c^2}{T}\;, \tag{31}$$

where E denotes the ratio of the relative magnitude of the centrifugal force to the surface tension and B the so-called Bond number.

The number E is directly related to the bubble-shaped factor K. Their relation is shown in Figure 2.

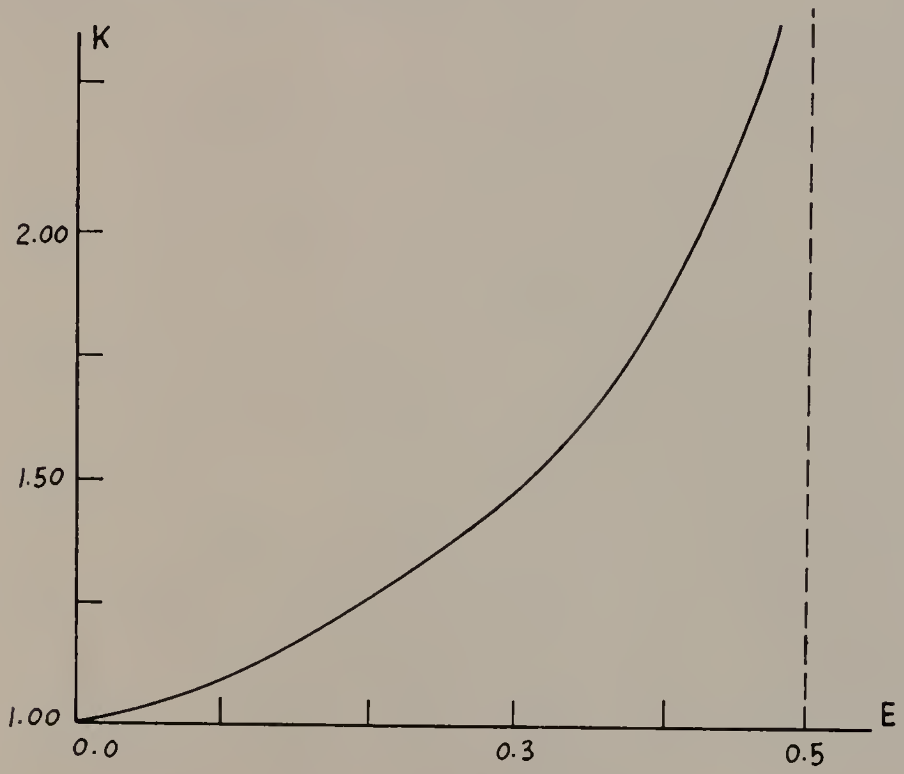

Figure 2. The shape factor K versus the dimensionless number E. For the limiting case E = 0.5, the bubble becomes an infinite cylindrical tube.

Then, finally, the dynamic boundary condition can be written as

$$8E\left\{\frac{a^2}{4}\alpha_0\alpha_1\cdot(1-\beta^2)+p_0\right\}+B\eta e^{i(\Omega t+\theta)}=-(J^*-J)\ , \tag{32}$$

with p_o as the perturbation pressure to be evaluated at the interface.

The change of curvature $J^* - J$ is a lengthy expression in terms of β, θ, $\alpha_1(\beta,\theta)$, and the first and second derivatives of $\alpha_1(\beta,\theta)$ with respect to β. The procedure of calculation follows the method outlined in Struik [5].

All the energy transfer and other important dynamical effects are contained in the boundary conditions at the free interface. The eigenfrequencies are determined by this boundary condition.

CHAPTER III

SOLUTION FOR THE ELLIPTIC CASE

3.1 Representation of the solution of the field equation

Since the Laplace equation (14) is separable in the prolate spheroidal coordinates, the perturbation pressure may be written as

$$P(\alpha,\beta,\theta) = \Lambda(\alpha)\, M(\beta)\, N(\theta)\,. \tag{33}$$

After a few steps of standard computation the solution of the field equation is obtained in the following form:

$$P(\alpha,\beta,\theta) = \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \left(A_{mn} P_n^m(\alpha) + B_{mn} Q_n^m(\alpha) \right) P_n^m(\beta)\, e^{im\theta}\,. \tag{34}$$

In the above representation, the functions $P_n^m(\alpha)$, $P_n^m(\beta)$, and $Q_n^m(\alpha)$ are the Legendre associated functions of the first and the second kind, respectively. The coefficients A_{mn} and B_{mn} are constants to be determined by the boundary conditions.

The dependence on the variable θ is given by the terms $e^{im\theta}$, where the number m is an integer which takes the values 0, 1, 2, ... Physically, these are the fundamental modes as viewed on each of the circular cross sections transverse to the axis of rotation. The case $m = 0$ indicates that each circular cross section of the bubble expands or contracts radially with the same amount for all θ . The case $m = 1$

shows that each circular cross section moves as a whole transverse to the axis of rotation. In this case the dynamic effects are asymmetric. If the perturbation pressure is integrated along any concentric circular path inside the fluid domain, the result is non-zero. There is a net resultant force acting on the fluid body bounded by this path. On the other hand, if the acceleration field, induced by the rotation, is symmetric, the bubble is freely floating inside the fluid and hence the surface tension offers no resistance to an asymmetric disturbance, except locally for a small portion of the entire interface. Had there been no external force field acting on this system, there would be no restoring force to sustain such an oscillation. Hence such an oscillation is possible only if some transverse inertia force field is present. In an inertial frame of reference a force field realizable by this mechanical system has always a relative frequency of oscillation less than two. Therefore for this mode of oscillation the flow field is hyperbolic in nature and does not belong to the present elliptic case. For the case $m = 2$, the bubble undergoes a breathing motion. For higher modes the motion of the fluid is of the same nature as for $m = 2$.

3.2 Matching the exterior boundary condition

The solution of the Laplace equation as represented in the prolate spheroidal coordinates forms a complete system of eigenfunctions at the bubble interface $\alpha = \alpha_0$. At the rigid cylindrical tank wall, the corresponding sequence of functions, as given by this representation, is, however, not complete. It is incapable of satisfying a boundary condition in general at the exterior boundary. Nevertheless, it is possible to match a homogeneous boundary condition at this boundary by means of a method described in this section.

It may be found also from the computation of the difference of curvature, $J^* - J$, that there is no interaction between the various modes of $e^{im\theta}$ due to the surface tension effect. Hence, each mode of different values of m can be handled separately. From the discussions of the previous section, only the cases where $m \geq 2$ are considered. Now the summation with respect to m may be dropped from the representation of the perturbation pressure. Then the number m remaining in the expressions in question acts as a parameter rather than an index.

In the following a system of eigenfunctions will be obtained in the cylindrical coordinate system such that:

a) The eigenfunctions are harmonic inside the tank;

b) Each eigenfunction satisfies part of the required boundary conditions at the rigid wall of the tank;

c) The system of eigenfunctions is complete over the entire exterior boundary, i.e., the cylindrical wall and the plane end disks.

Let it be required that the scalar product formed between P and the first N eigenfunctions over the entire surface of the exterior boundary are all zero, and further let N pass to infinity. Since the system of eigenfunctions is complete, the perturbation pressure satisfies the homogeneous boundary condition in the mean at the exterior boundary.

The Laplace equation as written in the cylindrical coordinate system is

$$\frac{\partial^2 \ell}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial \ell}{\partial \eta} + \frac{1}{\eta^2} \frac{\partial^2 \ell}{\partial \theta^2} + \frac{\partial^2 \ell}{\partial \zeta^2} = 0 \quad . \qquad (35)$$

Let P be represented as

$$P(\eta,\theta,\mu) = U(\eta)\,V(\theta)\,W(\mu)\,. \tag{36}$$

With this scheme of separation of variables, the following system of uncoupled ordinary differential equations is obtained

$$\left\{\begin{aligned} &\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dU}{d\eta}\right) - \left(\lambda^2 + \frac{m^2}{\eta^2}\right)U = 0\,, \\ &\frac{d^2V}{d\theta^2} + m^2 V = 0\,, \\ &\frac{d^2W}{d\mu^2} + \lambda^2 W = 0. \end{aligned}\right. \tag{37}$$

The eigenvalues for the parameter m are naturally determined as the sequence of integers 0, 1, 2, ... The eigenvalues of the parameter λ are obtained separately for the case $\lambda^2 > 0$ and the case $\lambda^2 < 0$. These two cases are different in character. When the parameter m is not zero, the value $\lambda = 0$ is not an eigenvalue for this system of differential equations and boundary conditions.

(A) The case $\lambda^2 > 0$

Under the requirement that the pressure field is symmetric with respect to μ and that the solution should be regular inside the tank, the solution of the Laplace equation (35) can be represented as

$$P(\eta,\theta,\mu) = \sum_{n=1}^{\infty} a_n I_m(\lambda_n\eta)\cos(\lambda_n\mu)\,e^{i(\omega t + m\theta)}\,, \tag{38}$$

where I_m is the modified Bessel function of the first kind and the a_n's are constants. The eigenfunctions must fulfill the boundary condition

$$(\omega^2-4)\,w = i\omega\frac{\partial \ell}{\partial \mu} = 0 \quad \text{at} \quad \mu = \ell^* \quad ,$$

i.e., at the plane end disks of the tank, where $\ell^* = \dfrac{\omega \ell}{\sqrt{\omega^2-4}}$.

Hence

$$\sum_{n=1}^{\infty} a_n(-i\omega\lambda_n) I_m(\lambda_n\eta) \sin(\lambda_n\mu)\cdot e^{i(\omega t+\theta)} = 0 \quad . \tag{39}$$

In order to have a system of nontrivial solutions, it is necessary that

$$\sin(\lambda_n\mu)\Big|_{\pm \ell^*} = 0 \; . \tag{40}$$

Hence the eigenvalues are determined by

$$\lambda_n = \frac{n\pi}{\ell^*} \qquad n = 1,2,3,\ \ldots\ . \tag{41}$$

This means physically that the normal velocity at the ends of the tank vanishes. The cylindrical boundary is left unrestricted and is free to oscillate. The normal velocity at the free cylindrical surface is given by

$$(\omega^2-4)\,u = \sum_{n=1}^{\infty} a_n(i\lambda_n)\Big\{\frac{\omega}{2}\big(I_{m-1}(\lambda_n\eta_0)+I_{m+1}(\lambda_n\eta_0)\big) + \\ +\big(I_{m-1}(\lambda_n\eta_0)-I_{m+1}(\lambda_n\eta_0)\big)\Big\}\cos(\lambda_n\mu)e^{i(\omega t+m\theta)} . \tag{42}$$

This sequence of eigenfunctions, representing the normal velocity at the entire exterior boundary, has the following properties:

a) The normal velocity at the ends of the tank vanishes;

b) The eigenfunctions are orthogonal to each other over the entire surface of the exterior boundary;

c) The sequence of eigenfunctions is complete when its domain is restricted to the cylindrical portion of the exterior boundary.

(B) The case $\lambda^2 < 0$

The main purpose for constructing these sequences of eigenfunctions is that the normal velocity representation, derived from the eigenfunctions of the perturbation pressure at the entire exterior boundary, form a complete system. Hence the choice of the boundary condition for the eigenfunctions is more or less free. In the present case there is no straightforward boundary condition that can be prescribed such as in case (A). The following boundary condition has been selected to start with

$$\left.\frac{\partial w}{\partial \eta}\right|_{\eta=\eta_0} = 0 \; . \tag{43}$$

The reasons for choosing this boundary condition are the following:

a) This is the weakest boundary condition which allows a choice among a variety of other conditions;

b) The eigenfunctions thus obtained do not depend on the parameter ω;

c) After orthogonalization with respect to the first sequence of eigenfunctions as obtained in the case (A), each of the eigenfunctions obtained in this case will satisfy the homogeneous boundary condition at the cylindrical boundary in the mean;

d) The normal velocity representation at the exterior boundary, corresponding to the eigenfunctions, forms a sequence of functions which are orthogonal and complete over the domain restricted to the ends of the tank.

In the present case the representation of the function P is given by

$$\mathcal{P}(\eta,\theta,\mu) = \sum_{n=1}^{\infty} b_n J_m(\lambda_n\eta)\cosh(\lambda_n\mu)\cdot e^{i(\omega t+m\theta)}, \tag{44}$$

where J_m is the Bessel function of order m and the b_n's are constants. The corresponding velocity components u and w are

$$(\omega^2-4)u = \sum_{n=1}^{\infty} b_n(i\lambda_n)\left\{\frac{\omega}{2}J_m'(\lambda_n\eta) + \frac{2m}{\lambda_n\eta}J_m(\lambda_n\eta)\right\}\cosh(\lambda_n\mu)e^{i(\omega t+m\theta)}, \tag{45}$$

$$(\omega^2-4)w = \sum_{n=1}^{\infty} b_n(i\lambda_n\omega)J_m(\lambda_n\eta)\sinh(\lambda_n\mu)e^{i(\omega t+m\theta)}, \tag{46}$$

and

$$(\omega^2-4)\frac{\partial w}{\partial\eta} = \sum_{n=1}^{\infty} b_n(i\lambda_n^2\omega)J_m'(\lambda_n\eta)\sinh(\lambda_n\mu)e^{i(\omega t+m\theta)}. \tag{47}$$

In order to satisfy the prescribed boundary condition it is necessary to put

$$J_m'(\lambda_n\eta_o) = \left.\frac{d\,J_m(\lambda_n\eta)}{d\eta}\right|_{\eta=\eta_o} = 0\ . \tag{48}$$

Hence the eigenvalues are the zeroes of the function $J_m'(\lambda_n\eta_o)$.

Thus the two sequences of eigenfunctions are completely defined. When the proper values of the coefficients a_n and b_n are given, these eigenfunctions, evaluated at the exterior boundary, can be written as:

Sequence (A)

$$\begin{cases} \phi_n = \cos(\lambda_n \mu) & \text{at the cylindrical wall,} \\[2ex] \phi_n = \dfrac{I_m(\lambda_n \eta)}{I_m(\lambda_n \eta_0)} & \text{at the plane end disks.} \end{cases} \tag{49}$$

Sequence (B)

$$\begin{cases} \psi_n = \dfrac{J_m(\lambda_n \eta_0)}{\cosh(\lambda_n l^*)} \cosh(\lambda_n \mu) & \text{at the cylindrical wall,} \\[2ex] \psi_n = J_m(\lambda_n \eta) & \text{at the plane end disks.} \end{cases} \tag{50}$$

with $n = 1, 2, \ldots$.

It is possible to put these two sequences of eigenfunctions together in order to form a single sequence of eigenfunctions such that the normal velocity representation given by these eigenfunctions forms a complete system of functions at the exterior boundary. Now there arises the question as to how the functions in these two sequences are to be denumerated into a single sequence of eigenfunctions. In the present study their relative importance is pointed out by the following consideration.

By returning to the prolate spheroidal coordinates, the zeroes of the function $P_n^m(\beta)$ are uniformly distributed along the φ-variable where φ is defined as $\cos\varphi = \beta$. In the three-dimensional configuration, the perturbation pressure would vanish along some conical surface $\beta = \text{const.}$ Let the coordinate surface passing the intersections of the cylinder with the end disks be denoted by $\beta = \pm \beta_0$.

For sufficiently large numbers of n, the number of waves of the function $P_n^m(\beta)$ outside the hyperboloids and the number of waves inside approaches a constant ratio ν, given by

$$\nu = \frac{\pi/2 - \varphi_0}{\varphi_0} , \tag{51}$$

where $\cos \varphi_o = \beta_o$, and $\varphi_o < \frac{\pi}{2}$.

For example:

when $\varphi_o = 30^o$, then $\nu = 2.000$.

Now the relative importance of the boundary condition at the cylindrical wall and the boundary condition at the end disks of the bubble oscillation problem is assumed to follow the same ratio ν. Then the new sequence of eigenfunctions is denumerated from the sequence (A) and the sequence (B) in such a way that in the first N terms of the new sequence, the ratio of the members from sequence (A) to the members from the sequence (B), shall always be the rational number closest to ν.

For the above mentioned example, the denumeration of the new sequence of functions $\{\Phi_n\}$ is

$$\Phi_1 = \phi_1 ,$$
$$\Phi_2 = \psi_1 ,$$
$$\Phi_3 = \phi_2 ,$$
$$\Phi_4 = \phi_3 ,$$

etc.

As said before, the normal velocity representation, given by the eigenfunctions of the sequence (A), forms a complete sequence of functions at the cylindrical portion of the boundary and vanishes at the disk ends, while the normal velocity representation, given by the eigenfunctions of the sequence (B) forms a complete system of functions at the end disks and, moreover, is a set of monotone functions, which vanish very rapidly, as n becomes large, over the half lengths of the cylindrical portion of the boundary. In the new sequence $\{ \Phi_n \}$, the members from the sequence (A) and the sequence (B) approach a constant ratio. It is easy to verify that the new sequence of eigenfunctions gives a complete system of functions for the normal velocity representation of the exterior boundary.

The eigenfunctions in the sequence $\{ \Phi_n \}$ can be orthogonalized by using the Hilbert-Schmidt procedure. In the present study, only the homogeneous boundary condition has to be satisfied. The sequence $\{ \Phi_n \}$ can be used directly. The results obtained are equivalent to those obtained by using the orthogonalized sequence of eigenfunctions.

Now the equivalent statement of the homogeneous boundary condition is

$$\oint_{\Sigma} \Phi_i(\eta,\theta,\mu) P(\alpha,\beta,\theta)\, d\sigma = 0, \quad n = 1, 2, 3, \dots , \tag{52}$$

where Σ denotes the entire surface of the tank.

In the above integrals, the function P and the eigenfunctions $\{ \Phi_n \}$ are expressed analytically in two different coordinate systems. Hence, these integrals can be handled conveniently only in a numerical fashion.

3.3 The dynamic (interior) boundary condition

Since various modes of different values of m are not coupled, the interior boundary condition (32) for $m \geq 2$ will reduce to

$$8E\left\{\frac{a^2}{4}\alpha_0\alpha_1(1-\beta^2) + p_0\right\} = -(J^* - J) \; . \qquad (53)$$

It should be noted that the term $B\eta e^{i(\Omega t + \theta)}$ in equation (32) belongs to the case $m = 1$. In equation (53) the perturbation pressure at the interface p_o, the difference of curvatures, $J^* - J$, and the perturbation of the bubble shape $\alpha_1(\beta,\theta)$, can all be expressed in terms of a series of the associated Legendre functions with unknown constant coefficients. The constant ω is the eigenfrequency parameter. For p_o we find

$$p_0 = \sum_{n=m}^{\infty} \left(A_{mn} P_n^m(\alpha_0) + B_{mn} Q_n^m(\alpha_0)\right) P_n^m(\beta)\, e^{i(\omega t + m\theta)} \; . \qquad (54)$$

The dynamic boundary condition is evaluated at the original equilibrium boundary of the bubble, $\alpha = \alpha_o$, thus this boundary condition is, apart from the factor $e^{i(\omega t+m\theta)}$, a function of β only. After some lengthy computations, for the quantities $J^* - J$, $\alpha_1(\beta)$, $\frac{d\alpha_1(\beta)}{d\beta}$, and $\frac{d^2\alpha_1(\beta)}{d\beta^2}$, there results

$$\begin{aligned} J^* - J = {} & \frac{f_c e^{i(\omega t + m\theta)}}{K^3(\alpha_0^2-1)^3}\left(1 + 1.725\frac{K^2-1}{K^2}\beta^2\right)\Big\{\left(a_0^{(1)} + a_2^{(1)}\beta^2\right)\alpha_1(\beta) + \\ & + \frac{m^2\beta^2}{(1-\beta^2)}(\alpha_0^2-1)(\alpha_0^2-\beta^2)\alpha_1(\beta) + \left(a_0^{(2)} + a_2^{(2)}\beta^2\right)\beta\frac{d\alpha_1(\beta)}{d\beta} + \\ & + (\alpha_0^2-\beta^2)(1-\beta^2)(\alpha_0^2-1)\frac{d^2\alpha_1(\beta)}{d\beta^2}\Big\} + \\ & + \frac{f_c e^{i(\omega t + m\theta)}}{K^5(\alpha_0^2-1)^5}\left(1 + 3.625\frac{K^2-1}{K^2}\beta^2\right)\cdot\Big\{\left(a_0^{(3)} + a_2^{(3)}\beta^2 + a_4^{(3)}\beta^4\right)\alpha_1(\beta) + \qquad (55) \\ & + \left(a_0^{(4)} + a_2^{(4)}\beta^2 + a_4^{(4)}\beta^4\right)\beta\cdot\frac{d\alpha_1(\beta)}{d\beta}\Big\} \; , \end{aligned}$$

$$\alpha_1(\beta) = \frac{4 f_D}{a^2(\alpha^2-\beta^2)} \sum_{n=m}^{\infty} \left\{ k_P^{(n)}(\alpha_0; m, \omega) A_{mn} + k_Q^{(n)}(\alpha_0; m, \omega) B_{mn} \right\} P_n^m(\beta) , \tag{56}$$

$$\frac{d\alpha_1(\beta)}{d\beta} = \frac{4 f_D}{a^2(\alpha_0^2-\beta^2)^2(1-\beta^2)} \sum_{n=m}^{\infty} \left\{ k_P^{(n)}(\alpha_0; m, \omega) A_{mn} + k_Q^{(n)}(\alpha_0; m, \omega) B_{mn} \right\} \cdot$$

$$\cdot \left\{ (2+(n+1)\alpha_0^2 - (m+3)\beta^2)\beta P_n^m(\beta) - (n-m+1)(\alpha_0^2-\beta^2) P_{n+1}^m(\beta) \right\} , \tag{57}$$

$$\frac{d^2\alpha_1(\beta)}{d\beta^2} = \frac{4 f_D}{\alpha^2(\alpha_0^2-\beta^2)^3(1-\beta^2)^2} \sum_{n=m}^{\infty} \left\{ k_P^{(n)}(\alpha_0; m, \omega) A_{mn} + k_Q^{(n)}(\alpha_0; m, \omega) B_{mn} \right\}$$

$$\cdot \left\{ (a_0^{(5)} + a_2^{(5)}\beta^2 + a_4^{(5)}\beta^4 + a_6^{(5)}\beta^6) P_n^m(\beta) + (a_0^{(6)} + a_2^{(6)}\beta^2 + a_4^{(6)}\beta^4)\beta P_{n+1}^m(\beta) \right\} , \tag{58}$$

with

$$a_0^{(1)} = -(5+m^2+mK^2+5K^2-m)\alpha_0^4 + \{6+(m+8)K^2\}\alpha_0^2 - (3K^2+1) ,$$

$$a_2^{(1)} = (m+5)(K^2-1)\alpha_0^4 + \{6+m^2-(m+8)K^2\}\alpha_0^2 + (3K^2-1) ,$$

$$a_0^{(2)} = -(3K-1)(\alpha_0^2-1)^2 ,$$

$$a_2^{(2)} = 3(K^2-1)(\alpha_0^2-1)^2 ;$$

$$a_0^{(3)} = -2\alpha_0^2(\alpha_0^2-1)(2\alpha_0^2-1)^2 + f_A(\alpha_0^2-1)(2\alpha_0^2-1)(5\alpha_0^2-1) - 2f_A^2(\alpha_0^2-1)(2\alpha_0^2-1) ,$$

$$a_2^{(3)} = 4\alpha_0^2(\alpha_0^2-1)(2\alpha_0^2-1) - 2f_A\alpha_0^2(\alpha_0^2-1)(5\alpha_0^2-1) + 4f_A^2(\alpha_0^2-1)(2\alpha_0^2-1) ,$$

$$a_4^{(3)} = -2\alpha_0^2(\alpha_0^2-1) + f_A(\alpha_0^2-1)(5\alpha_0^2-1) - 2f_A^2(\alpha_0^2-1)(2\alpha_0^2-1) ,$$

$$a_0^{(4)} = 2f_A(\alpha_0^2-1)^2(2\alpha_0^2-1) - 2f_A^2(\alpha_0^2-1)^2 ,$$

$$a_2^{(4)} = -4f_A\alpha_0^2(\alpha_0^2-1)^2 + 4f_A^2(\alpha_0^2-1)^2 ,$$

$$a_4^{(4)} = 2f_A(1-f_A)(\alpha_0^2-1)^2 ,$$

$$a_0^{(5)} = -(n^2+n-m^2)\alpha_0^4 + 2\alpha_0^2 ,$$

$$a_2^{(5)} = 2(\alpha_0^4 - m^2\alpha_0^2 + 3) + 3\alpha_0^2(\alpha_0^2+2)\cdot n + \alpha_0^2(\alpha_0^2+2)\cdot n^2 ,$$

$$a_4^{(5)} = -(2\alpha_0^2+1)(n^2+5n) - 6\alpha_0^2 + m^2 - 16 ,$$

$$a_6^{(5)} = n^2 + 7n + 12 ,$$

$$a_0^{(6)} = -2(n-m+1)\alpha_0^2(\alpha_0^2+2) ,$$

$$a_2^{(6)} = 4(n-m+1)(2\alpha_0^2+1) ,$$

$$a_4^{(6)} = -6(n-m+1) ,$$

$$f_A = \alpha_0^2 - (\alpha_0^2-1)K^2 ,$$

$$f_C = \frac{K\sqrt{\alpha_0^2-1}}{\alpha_0}$$

$$f_D = \frac{1}{4}\left\{\frac{\alpha_0^2}{K^2(\alpha_0^2-1)} - 1\right\}$$

$$k_P^{(n)} = \left\{\frac{2m}{\omega} - (n+1)\right\}\alpha_0 P_n^m(\alpha_0) + (n-m+1)P_{n+1}^m(\alpha_0) ,$$

$$k_Q^{(n)} = \left\{\frac{2m}{\omega} - (n+1)\right\}\alpha_0 Q_n^m(\alpha_0) + (n-m+1)Q_{n+1}^m(\alpha_0) .$$

The factor $(1 - \beta^2)$ vanishes at $\beta = \pm 1$. However, the terms involving this factor in the denominator remain regular due to the fact that the associated functions $P_n^m(\beta)$ have a zero of the same or higher order at these points when $m \geq 2$. For the same reason it is permissible to multiply the equation of the boundary condition by a factor $(\alpha_0^2 - \beta^2)^2\cdot(1 - \beta^2)$. By doing so, the boundary condition is reduced to a summation of terms of $P_n^m(\beta)$ with polynomials of β as the coefficients. It appears in the form

$$\sum_{m=n}^{\infty} \left\{ \left[\mathcal{P}_n^{(1)}(\beta) P_n^m(\beta) + \mathcal{P}_n^{(2)}(\beta) P_{n+1}^m(\beta) \right] + \frac{1}{\omega} \left[\mathcal{P}_n^{(3)}(\beta) P_n^m(\beta) + \mathcal{P}_n^{(4)}(\beta) P_n^m(\beta) \right] \right\} = 0, \qquad (59)$$

where $\mathcal{P}_n^{(i)}(\beta)$, $i = 1,2,3,4$, are polynomials of β. These polynomials are obtained by substituting equations (54), (55), and (56) into equation (53) and multiplying the entire expression by a factor $(\alpha_o^2 - \beta^2)^2 \cdot (1 - \beta^2)$. The highest order terms in the polynomials are β^8. This boundary condition can be reduced further to a series of $P_n^m(\beta)$ with constant coefficients by means of the recurrence formula

$$\beta P_n^m(\beta) = \frac{1}{2n+1} \left\{ (n-m+1) P_{n+1}^m(\beta) + (n+m) P_{n-1}^m(\beta) \right\} . \qquad (60)$$

Then, finally, the dynamic boundary condition can be written in the form

$$\sum_{n=m}^{\infty} \sum_{K=n-8}^{K=n+8} \left\{ f_K^{(n)}(\alpha_o; m, \omega) A_{mK} + g_K^{(n)}(\alpha_o; m, \omega) B_{mk} \right\} P_n^m(\beta) = 0 , \qquad (61)$$

where the $f_K^{(n)}(\alpha_o;m,\omega)$, and $g_K^{(n)}(\alpha_o;m,\omega)$, are constants determined by α_o, m, n, and containing ω linearly as an eigenparameter. The coefficients of the polynomials $\mathcal{P}_n^{(i)}(\beta)$, as well as the constants $f_K^{(n)}(\alpha_o;m,\omega)$ and $g_K^{(n)}(\alpha_o;m,\omega)$, which follow from equation (59) by repeated application of equation (60), are obtained numerically for given values of m, α_o, and ω.*

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*They are generated by the computer program and will not be reproduced here.

A system of linear algebraic equations in terms of A_{mn} and B_{mn} can be obtained by forming the scalar products of this boundary condition and the eigenfunctions $P_n^m(\beta)$ for $n \geq m$. Together with the exterior boundary condition, there are sufficient algebraic equations to determine the eigenvalues of ω, and subsequently the eigensolutions of A_{mn} and B_{mn}, which determine the mode shape of the oscillations of the bubble interface.

In concluding this section, there is one remark which should be made concerning the effects of the rotation of the entire system and the surface tension of the oscillations of the bubble. In the final form of the interior boundary condition there is a second summation sign which sums over k for nine terms.* In an elementary oscillation problem, the second summation sign is not present. Each simple mode of the harmonic oscillations, as given by the eigenfunctions, is distinct, such as in the case of the vibrations of a string with fixed end points. In the present linearized theory of oscillations, the surface tension effect induces an interaction among nine simple modes of harmonic oscillations, while the rotational effect alone will induce an interaction among three consecutive simple modes only.

3.4 Numerical solution

If the series representation of the perturbation pressure is truncated to n terms only, the boundary conditions can be reduced to a system of linear algebraic equations in terms of the unknown constant coefficients, and the eigenparameter ω is contained in these equations. Since the oscillations are symmetric, only the terms with even number of $(n-m)$ are non-zero. There are only $(n-m) + 2$ unknown constants, A_{mn} and B_{mn}. Each of the boundary conditions will render $\frac{1}{2}(n-m) + 1$ linear equations of the unknown constants. The system of linear equations is determinate.

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*In the present case the oscillations are symmetric with respect to β. Hence only the eigenfunctions with even values of (n-m) have nonzero coefficients. Only these terms are counted.

The reduction of the boundary condition to a system of algebraic equations is given in the following.

The exterior boundary condition (52) yields

$$\sum_{K=m}^{m+n} \left\{ g^{*}_{ik} A_{mk} + h^{*}_{ik} B_{mk} \right\} = 0, \qquad i = 1, 2, \cdots, \tfrac{1}{2}(n-m)+1 \qquad (62)$$

where

$$g^{*}_{ik} = \oint_{\Sigma} \Phi_i(\eta, \theta, \mu)\, P^{m}_{K}(\alpha)\, P^{m}_{K}(\beta)\, d\sigma \quad ;$$

$$h^{*}_{ik} = \oint_{\Sigma} \Phi_i(\eta, \theta, \mu)\, Q^{m}_{k}(\alpha)\, P^{m}_{k}(\beta)\, d\sigma \quad .$$

while the interior boundary condition (61) becomes

$$\sum_{K=m}^{m+n} \left\{ f^{(i)}_{k}(\alpha_o; m, \omega) A_{mk} + g^{(i)}_{k}(\alpha_o; m.\omega) B_{mk} \right\} = 0, \quad i = m, m+2, \cdots, m+n \qquad (63)$$

where $f^{(i)}_{K}(\alpha_o; m, \omega)$ and $g^{(i)}_{K}(\alpha_o; m, \omega)$ are the same as before. Furthermore, the coefficients can be written as

$$f^{(i)}_{K}(\alpha_o; m, \omega) = a^{*(1)}_{ik} - \frac{1}{\omega} b^{*(1)}_{ik} \quad ;$$

$$g^{(i)}_{K}(\alpha_o, m, \omega) = a^{*(2)}_{ik} - \frac{1}{\omega} b^{*(2)}_{ik} \quad .$$

At this point it is convenient to introduce matrix notation to handle the system of algebraic equations. All the matrices, G, H, $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, defined below, are $\{\frac{1}{2}(n-m)+1\} \times \{\frac{1}{2}(n-m)+1\}$ square matrices. The vectors X and Y are two column matrices with $\frac{1}{2}(n-m)+1$ elements. They are

$$\begin{aligned}
&G = \{g_{ij}\}, \quad g_{ij} = g^{*}_{i\beta}, && B^{(1)} = \{b^{(1)}_{ij}\}, \quad b^{(1)}_{ij} = b^{*(1)}_{\alpha\beta}, \\
&H = \{h_{ij}\}, \quad h_{ij} = h^{*}_{i\beta}, && B^{(2)} = \{b^{(2)}_{ij}\}, \quad b^{(2)}_{ij} = b^{*(2)}_{\alpha\beta}, \\
&A^{(1)} = \{a^{(1)}_{ij}\}, \quad a^{(1)}_{ij} = a^{*(1)}_{\alpha\beta}, && X = \{x_i\}, \quad x_i = A_{m\alpha}, \\
&A^{(2)} = \{a^{(2)}_{ij}\}, \quad a^{(2)}_{ij} = a^{*(2)}_{\alpha\beta}, && Y = \{y_i\}, \quad y_i = B_{m\alpha}, \\
& && i,j = 1, 2, \ldots, \tfrac{1}{2}(m-n)+1, \\
& && \alpha = m + 2(i-1), \\
& && \beta = m + 2(j-1).
\end{aligned} \tag{64}$$

Then the exterior boundary condition together with the interior boundary condition can be written in the following concise form:

$$\left\{\left|\begin{array}{c|c} A^{(1)} & A^{(2)} \\ \hline G & H \end{array}\right| - \frac{1}{\omega}\left|\begin{array}{c|c} B^{(1)} & B^{(2)} \\ \hline 0 & 0 \end{array}\right|\right\}\left|\begin{array}{c} X \\ \hline Y \end{array}\right| = 0 . \tag{65}$$

This matrix equation can be solved, employing standard notation as follows:

First we have

$$GX + HY = 0, \tag{66}$$

which yields

$$X = - G^{-1}HY. \tag{67}$$

Further

$$(A^{(1)} - \frac{1}{\omega} B^{(1)})X + (A^{(2)} - \frac{1}{\omega} B^{(2)})Y = 0, \tag{68}$$

upon substituting for X, we have

$$(A^{(1)} - \frac{1}{\omega} B^{(1)})(-G^{-1}HY) + (A^{(2)} - \frac{1}{\omega} B^{(2)})Y = 0,$$

or

$$[(A^{(2)}-A^{(1)}G^{-1}H) - \frac{1}{\omega} (B^{(2)}-BG^{-1}H)]Y = 0. \tag{69}$$

Then with the definitions

$$R = A^{(2)}-A^{(1)}G^{-1}H \quad \text{and} \quad S = B^{(2)}-B^{(1)}G^{-1}H,$$

there results

$$(R - \frac{1}{\omega} S)Y = 0. \tag{70}$$

Equation (70) can be written as

$$(R - \frac{1}{\omega} S)S^{-1}SY = 0,$$

thus, finally

$$(RS^{-1} - \frac{1}{\omega} E)SY = 0, \tag{71}$$

where E is the unit matrix.

The eigenvalues $\frac{1}{\omega}$ and the eigenvectors $Z = (SY)$ can be found from the last equation. Then the eigenvectors can be expressed in terms of the original X and Y:

$$Z = SY,$$

yields

$$Y = S^{-1}Z, \tag{72}$$

and hence by equation (67)

$$X = -G^{-1}HY.$$

For any specific example, where dimensions of the tank, the density of the liquid, the constant speed of rotation, and hence the static equilibrium shape of the bubble are given, numerical results can easily be obtained by performing the computations according to this theory on a digital computer.

In the numerical computation the scale of the affine transformation

$$\frac{\omega}{\sqrt{\omega^2 - 4}} \qquad \text{in} \qquad \mu = \frac{\omega\zeta}{\sqrt{\omega^2 - 4}} ,$$

has to be chosen at the beginning. Then according to the obtained eigenvalues of ω this scale of the affine transformation can be computed employing the above given formula for μ. The new value of this scale factor is then used for the computation of a more accurate value of ω.

It is important to point out that a different scale factor exists for each different eigenvalue of ω. For very large values of ω the scale factor is close to 1.

3.5 Results and discussion

Numerical computations have been made for the case $m = 2$, which is the lowest circumferential mode under the scope of the elliptic case. The result shows that the first eigenfrequency above 2.000 corresponds always to the first mode of oscillation in the meridian plane. For any specific example we are able to compute the first and the second eigenfrequencies accurately. We may conclude that all the natural frequencies of the bubble oscillations are greater than two and their corresponding perturbation pressure field is elliptic in nature, since we have obtained the lower bound of these frequencies.

Some of the numerical results are presented in Table 1, Figure 3, and Figure 4. The computation is performed for a sequence of bubble shapes while the tank dimensions remain the same. The first and the second eigenfrequencies for each shape factor K are given numerically in Table 1. They are also plotted versus the shape factor K in Figure 3. Typical configurations for each mode in a meridian section are given in Figure 4. It is interesting to note that the excitation of the interface wave is much more pronounced in the neighborhood of the equator than in the pole regions of the bubble.

In the numerical computations n is taken as large as $m + 12$ such that up to seven non-zero terms in the series are included. At least five non-zero terms shall be taken in order to obtain an accurate result for the eigenfrequencies. This is not surprising because the interactions among the fundamental harmonic functions are very strong.

The third eigenfrequency may be obtained also. However, the accuracy is considerably less than the first two, due to the limited capacity of the computer.

TABLE 1

THE FIRST AND SECOND EIGENFREQUENCIES FOR BUBBLES WITH DIFFERENT SHAPE FACTORS IN A CYLINDRICAL TANK WITH FIXED DIMENSIONS

The shape factor K	The first eigenfrequency	The second eigenfrequency
1.100	2.296	2.409
1.200	2.401	2.558
1.400	2.516	2.780
1.600	2.620	2.977
1.800	2.754	3.235
2.000	2.941	3.605
2.200	3.168	4.007

The dimensions of the tank: $\begin{cases} \eta_o = 2.000, \\ \iota = 3.000. \end{cases}$

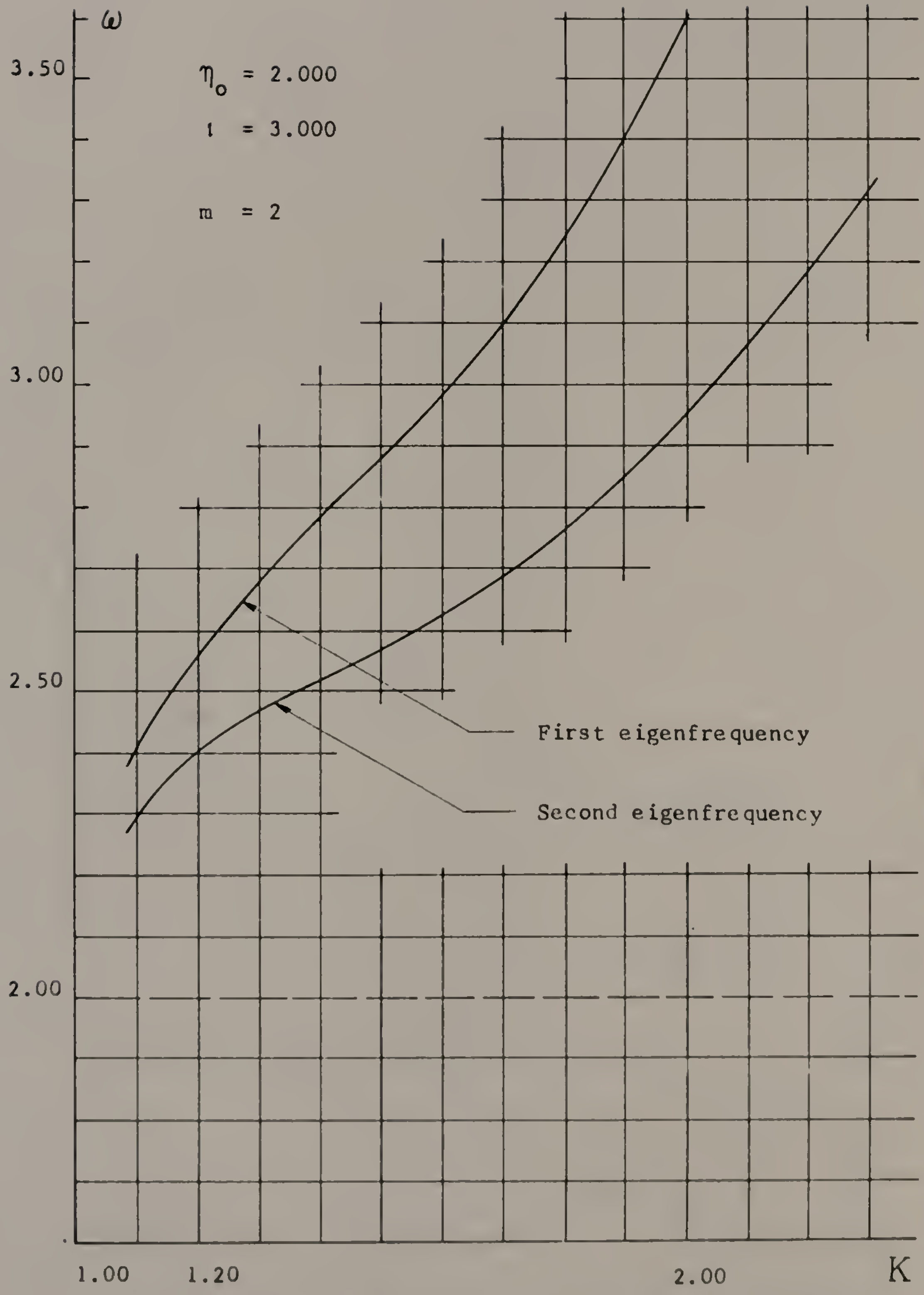


Figure 3. The first and second eigenfrequencies plotted versus the bubble shape factor K.

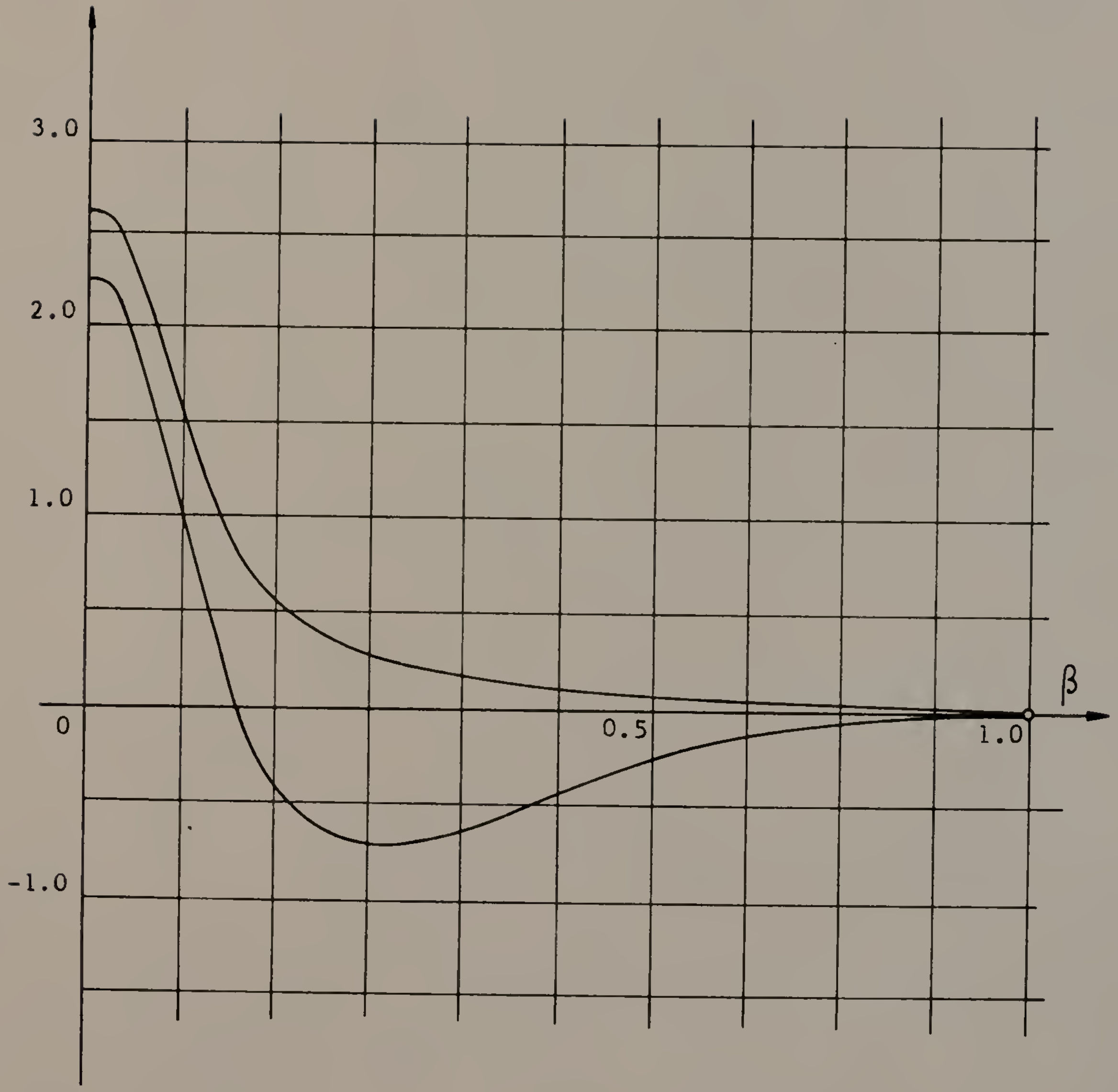


Figure 4. The typical mode shapes in the meridian plane corresponding to the first and the second eigenfrequencies of the oscillations of the bubble about its position of stable equilibrium.

CHAPTER IV

SOLUTION TO THE HYPERBOLIC CASE

4.1 Introduction

In this chapter the dynamical response of the rotating fluid system with respect to the perturbation due to a constant reduced gravity field transverse to the tank axis will be studied. From the discussion on page 18, we know that for the mode of oscillation $m = 1$, the motion has to be accompanied by an external force field. The governing equation is hyperbolic. Hence the method of analysis and the mathematical formulation of the problem are different from the elliptic case studied above.

For the physical conditions given above, the frequency of the perturbation ω relative to the rotating system is one. However, the analysis for this particular case, as persued in the following sections of this chapter, is valid also for any frequency in the range $0 < \omega < 2$.

The mathematical formulation here is written in cylindrical coordinates and the "characteristic coordinates." The latter will be defined in this chapter.

Now the side conditions for a well-posed problem in the sense of Hadamard are different for hyperbolic and elliptic equations. For the present problem, we have to prescribe kinematic boundary conditions at the cylindrical wall, the disk ends and the interface, as well as a dynamic condition at the interface. Such a system of boundary conditions is too stringent for the solution to exist. We shall see in section 4.4 that the boundary condition at the ends of the tank has to be relaxed. Hence we are considering a cylinder of infinite length instead of one with finite length.

Since the differential equation governing the pressure field is hyperbolic, there are real characteristic surfaces in the flow domain. Across these surfaces the normal derivatives may be discontinuous. Hence the velocity field will suffer a finite jump at the same location. The work by Oser [6] provides a good example.

4.2 Formulation of the problem

The field equation in cylindrical coordinates (η,θ,ζ) for the case $m = 1$ and $\omega = 1$ is given by

$$\frac{\partial^2 P}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial P}{\partial \eta} - \frac{1}{\eta^2}P - 3\frac{\partial^2 P}{\partial \zeta^2} = 0 \ .$$

The first order derivative term in this equation can be eliminated by means of the transformation

$$P(\eta,\theta,\zeta) = \eta^{-\frac{1}{2}}\,\Phi(\eta,\theta,\zeta) \cdot \tag{73}$$

Then the partial differential equation for Φ takes the form

$$\frac{\partial^2 \Phi}{\partial \eta^2} - 3\frac{\partial^2 \Phi}{\partial \zeta^2} = \frac{3}{4}\frac{1}{\eta^2}\Phi \ . \tag{74}$$

This transformation has been carried out for the sake of convenience regarding later numerical computations.

For the above hyperbolic equation (74), there are two families of real characteristic curves through each point of the (η,ζ) plane. Both of these families are straight lines.

These characteristics may be chosen as our coordinate system. The characteristic transformation in question is

$$\begin{cases} \sqrt{3}\,\eta + \zeta = 2\sqrt{3}\,\chi \quad , \\ \sqrt{3}\,\eta - \zeta = 2\sqrt{3}\,\tau \quad ; \end{cases} \tag{75}$$

where χ = const. and τ = const. are the characteristic lines from each of the two families.

Along the characteristic directions, the transformation does not determine the second derivatives uniquely. Hence across a characteristic line, the normal derivative may suffer a jump. However, the function ϕ is supposed to be continuous for a hydrodynamical problem.

Through the transformation (75), the hyperbolic differential equation (74) turns into the normal form:

$$\frac{\partial^2 \Phi}{\partial \chi \partial \tau} = \frac{3}{4}\,\frac{\Phi}{(\chi+\tau)^2} \quad . \tag{76}$$

For this equation two types of problems can be posed.

(A) The Cauchy problem

The initial conditions are prescribed on a curve PQ which is nowhere tangent to the characteristic directions. Thus the values of the function ϕ and its normal derivative on this curve are given. A solution exists and is uniquely determined in the triangular domain PQR (Figure 5).

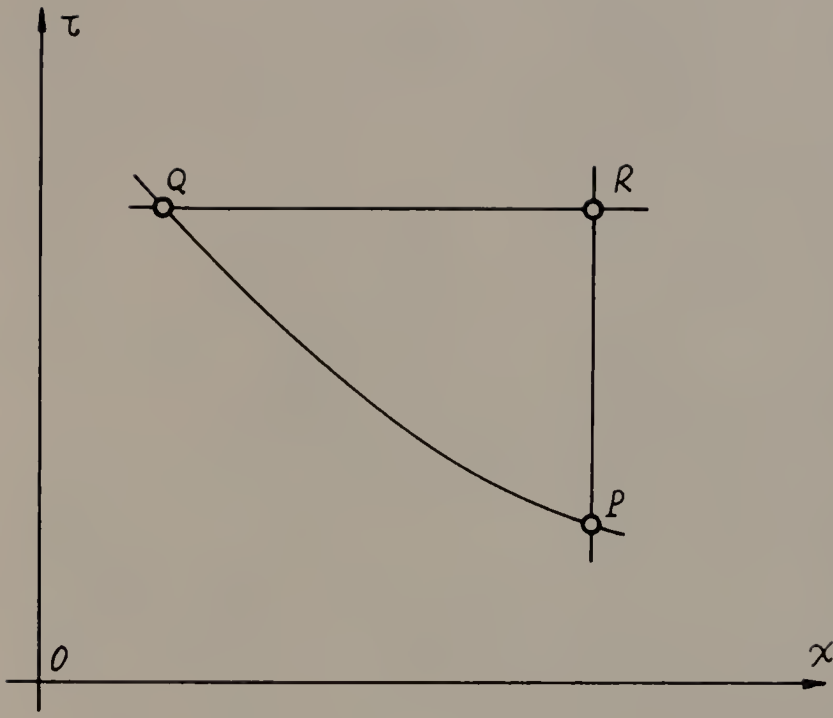


Figure 5. A sketch for the Cauchy problem.

The entire system of differential equations and the initial conditions can be combined into the following integro-differential equation (Garabedian [7]):

$$\Phi(R) = \frac{\Phi(P)+\Phi(Q)}{2} + \frac{1}{2}\int_P^Q \left(\frac{\partial\Phi}{\partial x}\,dx - \frac{\partial\Phi}{\partial\tau}\,d\tau\right) + \oiint_{PQR} \frac{3\,\Phi}{4(x+\tau)^2}\,dx\,d\tau \;. \qquad (77)$$

This equation can be solved by an iteration technique. This process is proved to be convergent. The solution exists in the large when the equation is linear, as is the present case.

(B) The Goursat problem

We may also prescribe one boundary condition on the ordinary (non-characteristic) curve and one on a characteristic line. If the value of Φ is prescribed on these curves, the problem is called the Goursat problem. A solution exists also and is uniquely determined in the triangular domain OTQ (Figure 6).

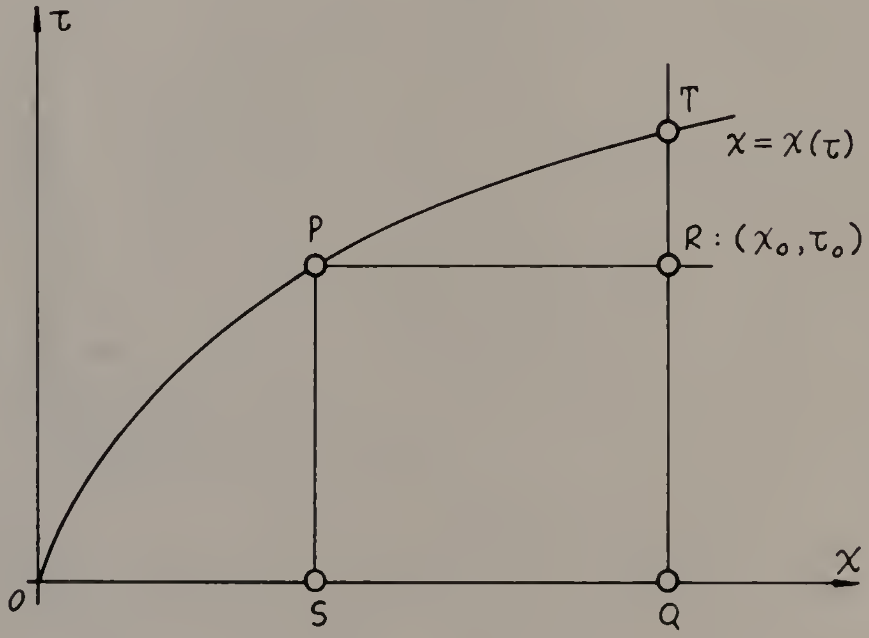


Figure 6. A sketch for the Goursat problem.

This problem can be formulated as an integral equation, namely,

$$\Phi(R) = \Phi(P) + \Phi(Q) - \Phi(S) + \iint_{PSQR} \frac{3\Phi}{4(\tau+\chi)^2}\, d\chi\, d\tau \quad (78)$$

This equation can be solved by means of Picard's method of successive approximations as in (A). When the boundary condition on the ordinary curve is replaced by a mixed type condition

$$\frac{\partial \Phi}{\partial \tau} + a(\tau)\frac{\partial \Phi}{\partial \chi} + b(\tau)\,\Phi = 0 \quad , \tag{79}$$

the existence and uniqueness of the solution remain unaltered. The problem is well posed. However, the integro-differential equation is somewhat complicated. Let us denote the ordinary curve by

$$\chi = \chi(\tau) \quad .$$

The function $\chi(\tau)$ is strictly monotonic. Then the integro-differential equation in question can be written in the form:

$$\Phi(R) = \Phi(Q) - \Phi(S) - e^{-I(\tau_0)}\int_0^{\tau_0}\left[b(\tau)\Phi(\chi) + a(\tau)\frac{\partial \Phi}{\partial \chi}\right]_{\chi=\chi(\tau)} \cdot e^{I(\tau)}d\tau + \oiint_{RPSQ}\frac{3\,\Phi}{4(\chi+\tau)^2}\,d\chi\, d\tau \, ,$$

$$I(\tau) = \int_0^{\tau} b(\tilde{\tau})\,d\tilde{\tau} \quad . \tag{80}$$

The solution of the Goursat problem for a linear equation exists in the large, as in case (A).

Numerical computations have been performed according to the above given integro-differential equations.

4.3 The boundary conditions

The boundary conditions prescribed for this problem have been derived in Chapter II. They are given by equations (15), (16), (24), and (32). They are listed below as rewritten in the cylindrical coordinates (η, θ, ζ).

(A) The kinematic boundary conditions at the walls of the tank:

$$\frac{\partial \Phi}{\partial \eta} + \frac{1.5}{\eta}\Phi = 0 \qquad \text{at the cylindrical wall,}$$
$$\frac{\partial \Phi}{\partial \zeta} = 0 \qquad \text{at the plane disk ends.} \tag{81}$$

(B) The kinematic boundary condition at the interface:

$$V_n - \delta = 0 \quad . \tag{82}$$

The normal velocity v_n and the normal displacement δ of the liquid-vapor interface are actually 90° out of phase when θ and t are taken into account.

(C) The dynamic boundary condition at the interface:

$$8E(\eta\delta_\eta + p_o) + B\eta = -(J^* - J) \cdot$$

where δ_η is the component of δ in the η-direction. This boundary condition, especially its right hand side, is very complicated. We shall replace it by two asymptotic representations. This approximation produces some quantitative errors, while it demonstrates, however, the qualitative natures of the problem much more clearly.

For a small displacement δ of the bubble interface, the change of curvature may be approximated by

$$J^* - J = \frac{d^2\delta}{ds^2} ,$$

where ds is a length element measured along the generator of the bubble interface.

In the neighborhood of the equator of the bubble, $d\zeta \simeq ds$. Hence we have the following asymptotic representation of the boundary condition:

$$8E\,(\eta\,\delta_\eta + p_o) + B\eta = -\frac{d^2\delta}{d\zeta^2} \quad . \tag{83}$$

In the neighborhood of the poles of the bubble the curvature of the bubble is pronounced. The geometry in this region is best described through the spheroidal coordinates defined in the previous chapters (Figure 7).

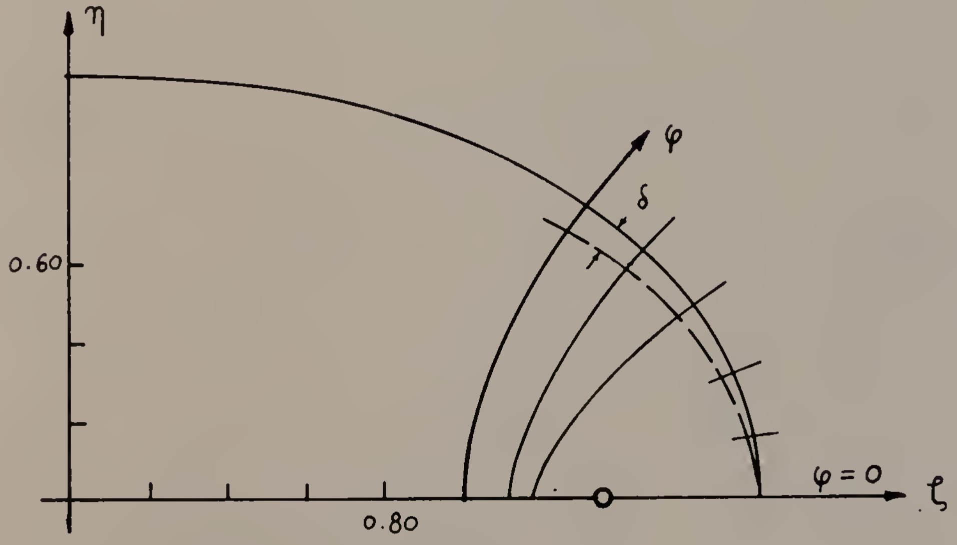


Figure 7. The coordinate system employed at the pole region of the bubble.

The formulas applicable to the present situation are summarized below:

$$\alpha_0 = K/\sqrt{K^2-1} \quad ,$$

$$\beta = \cos\varphi \quad ,$$

$$\eta = \sin\varphi \quad ,$$

$$\zeta = K\cos\varphi \quad ,$$

$$\frac{d\varphi}{ds} = 1/\sqrt{1+(K^2-1)\sin^2\varphi} \quad .$$

Near the pole the value of φ is small. The curvature may be written as:

$$\frac{d^2\delta}{ds^2} = \frac{d^2\delta}{d\varphi^2}\left(\frac{d\varphi}{ds}\right)^2 .$$

The η-component of the displacement δ, δ_η, is also very small in this region. It can be neglected. Hence we have another asymptotic representation for the dynamic boundary condition:

$$8E\,p_0 + B\sin\varphi = -\frac{d^2\delta}{d\varphi^2}\left(\frac{d\varphi}{ds}\right)^2 . \qquad (84)$$

The quantity $(\frac{d\varphi}{ds})$ approaches one as φ approaches zero.

The combination of the kinematic condition (82) and the dynamical condition (83) at the interface provides us only with one boundary condition for the field equation (76).

(D) The condition of symmetry at the equatorial plane

We assume that the flow field is symmetric with respect to the equatorial plane $\zeta = 0$. Hence we have the condition

$$\left.\frac{\partial\Phi}{\partial\zeta}\right|_{\zeta=0} = 0 \quad .$$

There are two more conditions for the determination of the flow field. These conditions will be introduced in the next section in connection with the construction of the solution to this problem.

4.4 Construction of the solution

For a hyperbolic equation discontinuities of the normal derivatives are admissible across a characteristic line. We shall construct a solution to this problem with all the possible discontinuities in mind. Figure 8 shows the division of the flow field into regions separated by characteristics where discontinuities may occur. In this figure the point G is located such that the bubble is tangent to the characteristic line.

As indicated in 4.3, we have to postulate two additional conditions to construct the flow field.

a) In the immediate neighborhood of the equatorial plane, the flow field is symmetric. The perturbation velocity normal to this plane vanishes. The bubble is tangent to a cylinder. Hence the perturbation pressure distribution given in Phillips [4] is exactly the same as the perturbation pressure distribution at the equatorial plane. In our notation there follows

$$\Phi\Big|_{\zeta=0} = -\frac{1}{2}\,\frac{B}{8E}\,\eta^{\frac{3}{2}}\left(\frac{3}{\eta^2}-\frac{1}{\eta_0^2}\right) . \tag{85}$$

b) The perturbation velocity at the axis of rotation is finite. From the expression of the perturbation velocity (6), we arrive at the condition

$$\Phi = 0 \quad .$$

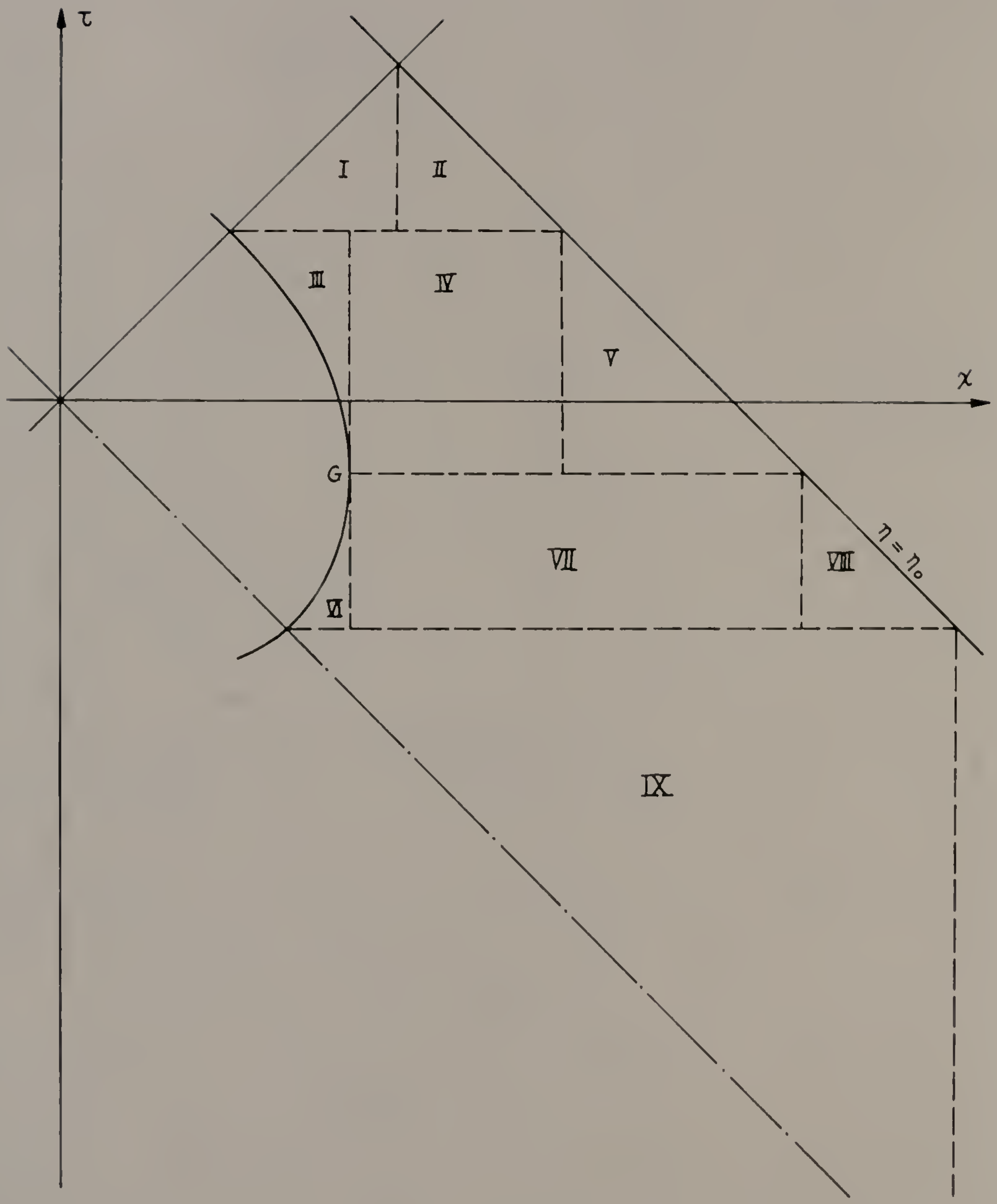


Figure 8. Division of the flow field into regions separated by characteristic lines where discontinuities may occur.

We can now proceed with the construction of the solution.

The condition 4.3(D) together with condition 4.4(a) form a set of initial conditions for the determination of the flow in the region I. Consequently, the generalized Goursat problems can be defined in the regions II through V successively. Their solutions are obtained by means of equation (78) or equation (80). In the region VI, the solution is obtained as follows.

The displacement of the bubble δ at $\varphi = 0$ vanishes for the geometrical compatibility requirement. In a neighborhood of the pole such that $\eta < \left| \frac{B}{8E} \right|$, both the inertia and the centrifugal field forces are smaller than the surface tension and the reduced forces by at least two orders of magnitude. Hence from equation (84) we have

$$\frac{d^2\delta}{d\varphi^2}\left(\frac{d\varphi}{ds}\right)^2 = - B \sin\varphi \quad ,$$

or,

$$\frac{d^2\delta}{d\varphi^2} = - B \left[1 + (K^2 - 1)\sin^2\varphi \right] \sin\varphi \quad ,$$

hence

$$\delta = B\left[\left(1 + \tfrac{2}{3}(K^2-1)\right)\sin\varphi + \tfrac{1}{3}\sin^3\varphi \right] + B\sigma_0\varphi \quad , \tag{86}$$

where the constant σ_o is determined by the condition that the displacement at G is continuous. By means of equations (82) and (84), the solution in region VI is determined.

When we proceed to construct the solution in further regions, we find that the problem is overdetermined when the boundary condition at the disk end of the tank is prescribed. Consequently a solution to the problem would not exist in the entire flow domain at all. This boundary has to be removed in order that a solution may exist. Once this boundary is removed, we see immediately that the solution extends to infinity. Physically it means that a steady state solution does not exist for a tank of finite length. The solution exists for a cylindrical tank of infinite length. In this case the perturbation propagates to infinity immediately. Such an unexpected phenomenon was found in a similar problem investigated by Benjamin and Barnard [8].

The solution in the regions VII, VIII, IX, ... , can easily be determined by a sequence of generalized Goursat problems. The perturbation field approaches zero as we extend the solution to infinity along the ζ-direction.

A numerical example is given in Figures 9 and 10.

4.5 Discussion

From the results obtained from this chapter, we have the following physical picture for the hyperbolic case.

The liquid-vapor system is stable with respect to the forced perturbation, since its dynamic response to the perturbation is an oscillation about the stationary equilibrium configuration with a small amplitude.

In the region RSGA (Figure 10), there is a strong exchange between the kinetic energy, the energy of the perturbation pressure field, and the centrifugal field effects. The effect of surface tension is negligible in this region. There is a strong secondary circulation induced by the disturbing forces.

The disturbances in the region below $\overrightarrow{AG}$ are small. Centrifugal and inertia effects are negligible in this region. The surface

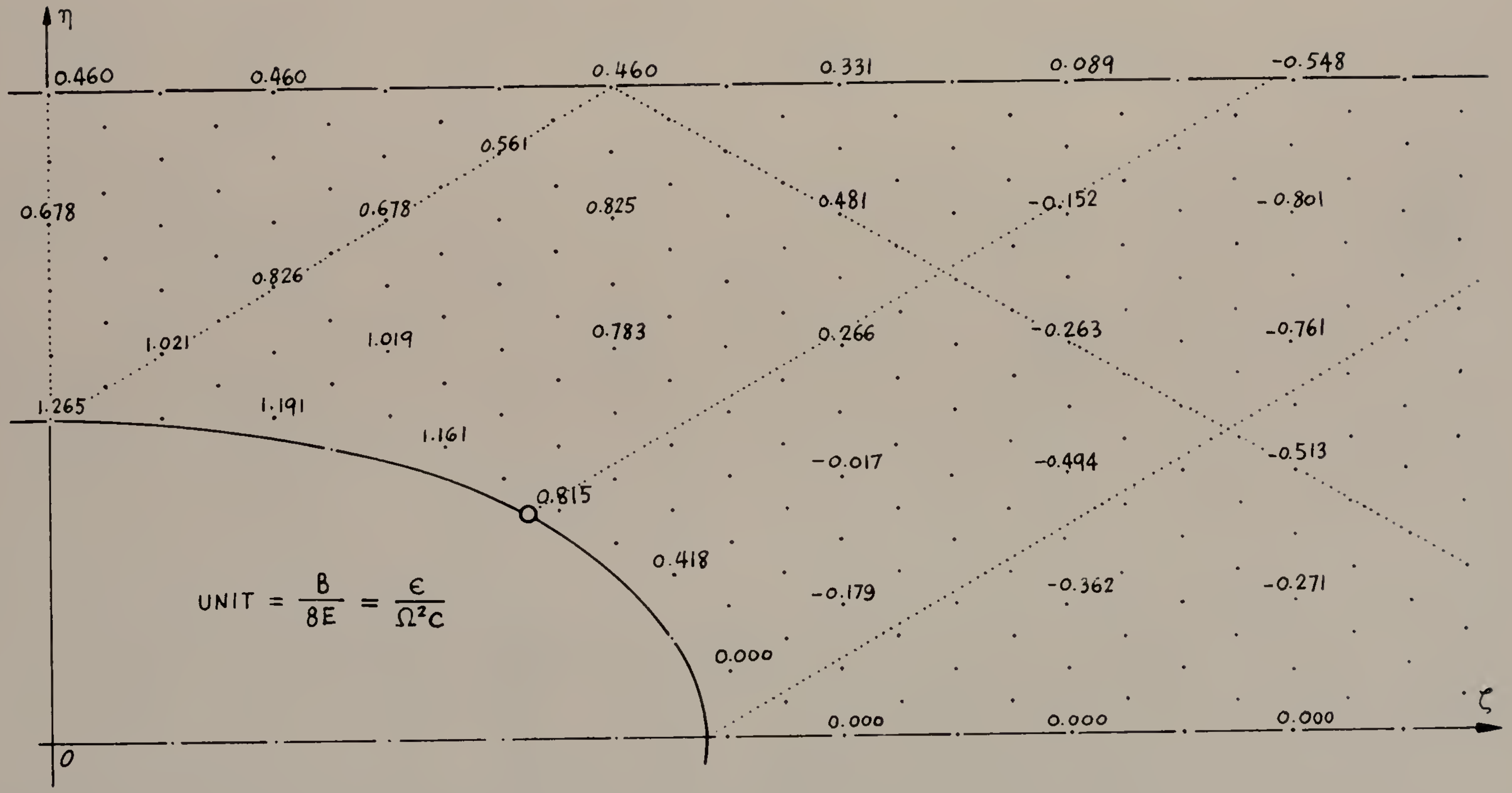


Figure 9. The perturbation pressure field for a tank-liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

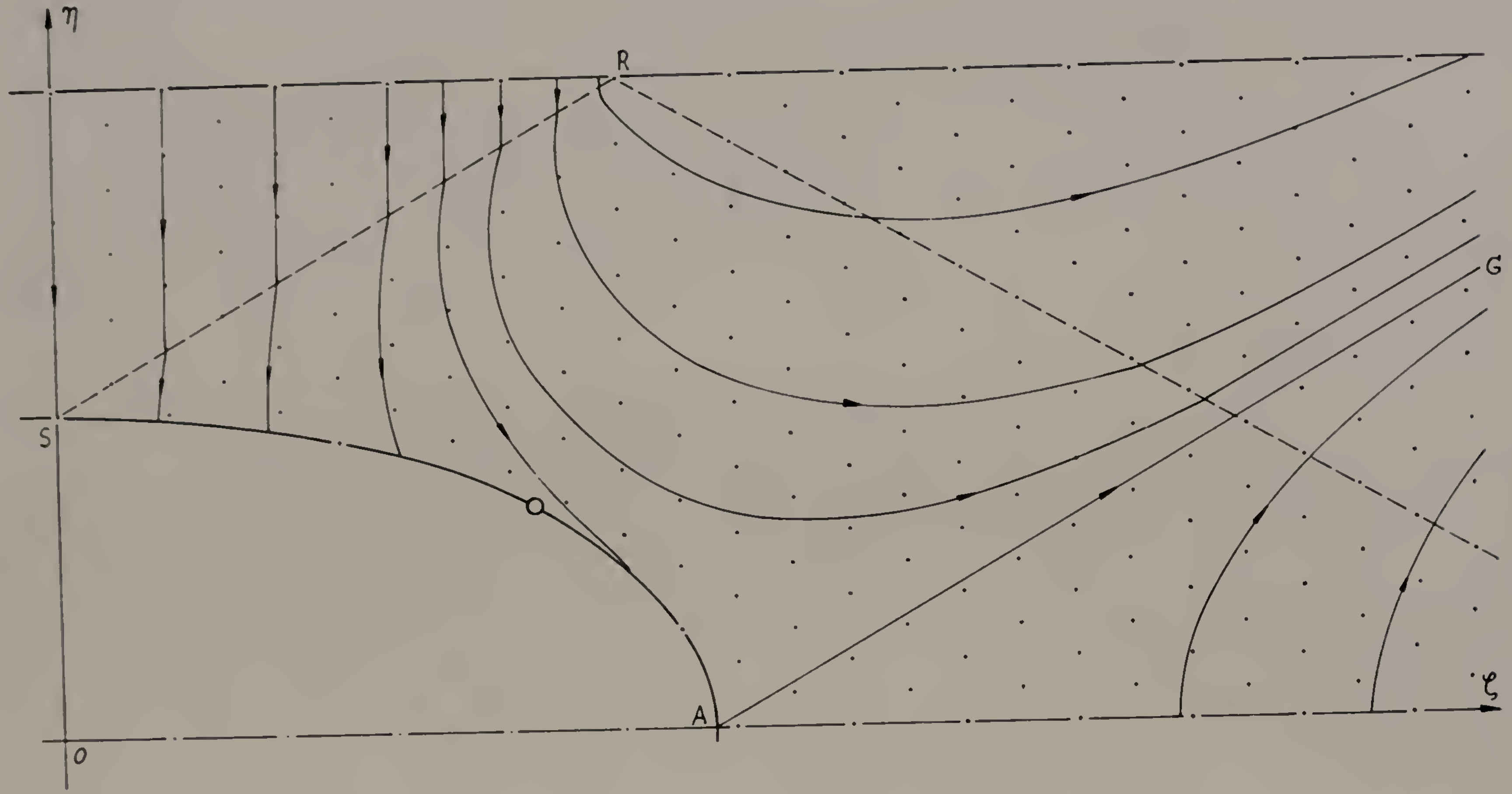


Figure 10. The perturbation velocity field for a tank-liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

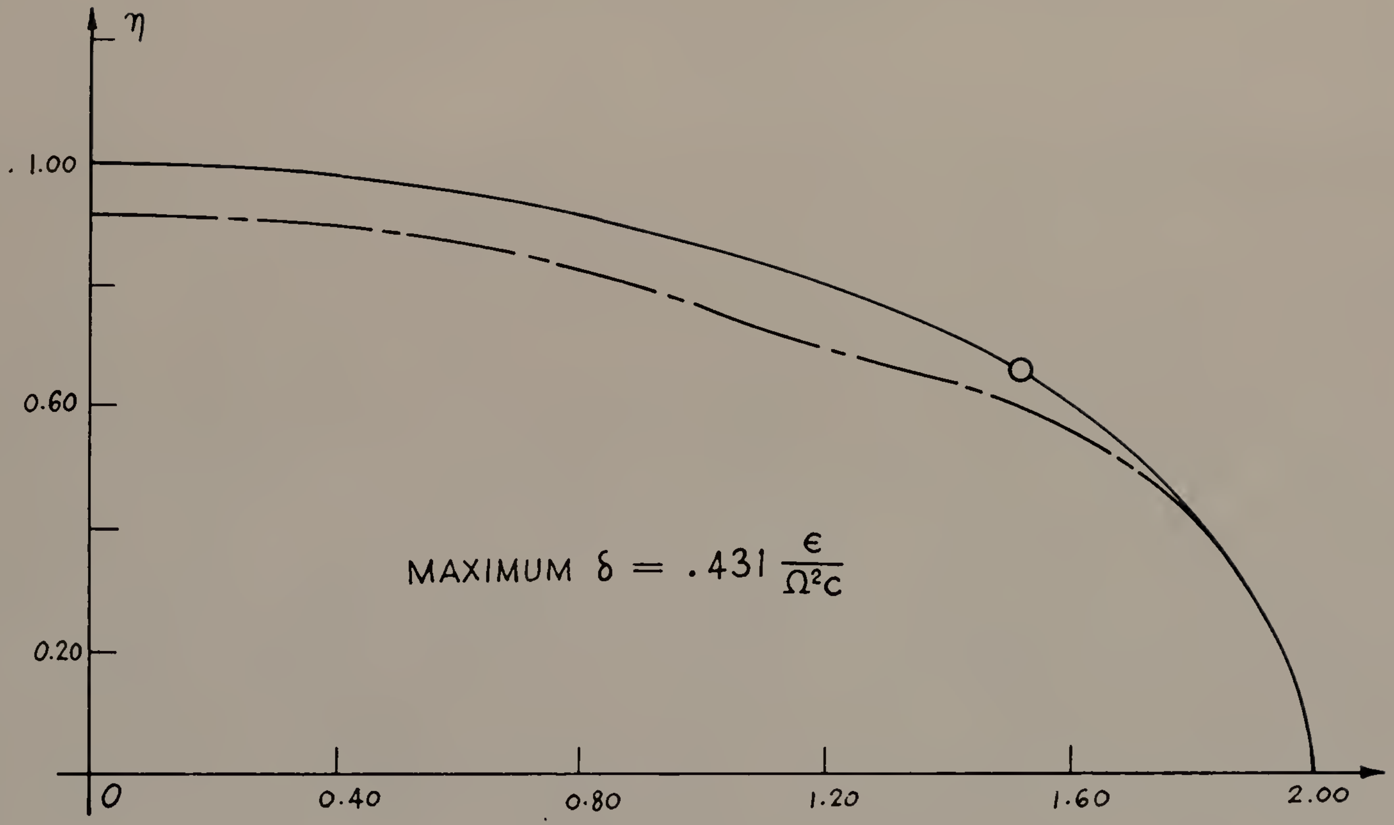


Figure 11. The deformed bubble shape for a tank-liquid system, rotating with a constant angular speed, under the influence of a transverse reduced gravity field.

tension provides the necessary adjustment to absorb the pressure field perturbations. In this region, the liquid particle performs mainly an oscillatory motion in the neighborhood of its own equilibrium position.

CHAPTER V

CONCLUSIONS

In the previous chapters we have determined the dynamic response of the rotating fluid system for the entire range of frequencies. In conclusion, we shall discuss here some of the physical implications of the results obtained above.

The slow rotation of the system with a constant angular speed has a profound effect on the dynamic response. For a rotating system, all the small oscillations of the liquid-vapor interface, or rather, of the entire liquid body, are stable. A small transverse disturbance to the system will induce one or several modes of oscillation about the stable equilibrium configuration.

On the other hand, under the influence of a transverse constant force field, the stability of the configuration of the system in the neighborhood of the equator of the bubble is ensured by the centrifugal pressure field and the inertia force produced by a small perturbation to the constant rotating base flow. The surface tension effect is negligible in this region. For a system without rotation, a disturbance containing such a force component is liable to excite instability.

Furthermore, for a real system viscosity effects are always present. For a system with rotation, a perturbation will induce some secondary circulation. The disturbance will be dissipated by means of the viscous mechanism. The surface tension effect alone is a two-dimensional mechanism. It can restore a disturbed system to an equilibrium configuration in a much longer period of time.

Finally, we shall present here an example to show the actual magnitude of various quantities characterizing a rotating tank-liquid system in low gravity environments.

Example

Given:

The radius of the tank	$R_o = 100$ cm ,
Semiminor axis of the undisturbed bubble	$C = 40$ cm ,
Bubble shape factor	$K = 2.00$,
The dimensionless number E corresponding to the bubble shape factor K	$E = 0.43$,
Density of the liquid	$\rho = 0.070$ g/cm^3 ,
Surface tension	$T = 2.00$ dyne/cm .

The liquid chosen in this example is liquid hydrogen. From the above given data, we may compute the corresponding constant angular speed of rotation Ω and the allowable magnitude of the transverse reduced gravity field:

$$\Omega = \sqrt{\frac{8ET}{\rho c^2}} = \sqrt{\frac{8 \times 0.43 \times 2.0}{0.07 \times (40)^3}} = \frac{1}{25} \text{ rad/sec} ,$$

$$B << 8E .$$

Assume that

$$\frac{B}{8E} = \frac{\epsilon}{\Omega^2 C} = 0.10 \; .$$

Then

$$\epsilon = 0.10 \times \Omega^2 \times C = \frac{1}{160} \; \mathrm{cm/sec} \; .$$

For this small acceleration the system can cover a distance of 100 feet in fifteen minutes. This would allow the system to perform a slow maneuver like changing the orientation of the tank, etc.

APPENDIXES

APPENDIX A

EXPRESSION OF THE PERTURBATION VELOCITY COMPONENT v_α IN THE PROLATE SPHEROIDAL COORDINATES (α, θ, β)

The expression of v_α may be derived from the known expressions of u and w in the deformed cylindrical coordinates (η, θ, μ),

$$\begin{cases} (\omega^2 - 4)\, u = i\omega \dfrac{\partial P}{\partial \eta} + \dfrac{2}{\eta} \dfrac{\partial P}{\partial \theta} \,, \\[2ex] (\omega^2 - 4)\, w = i\omega \dfrac{\partial P}{\partial \mu} \,, \end{cases} \tag{A1}$$

by means of the defined coordinate transformation relation, equation (13).

Let us define

$$\left| \frac{\partial(\alpha, \beta)}{\partial(\eta, \mu)} \right| = \begin{vmatrix} \dfrac{\partial \alpha}{\partial \eta} & \dfrac{\partial \beta}{\partial \eta} \\[2ex] \dfrac{\partial \alpha}{\partial \mu} & \dfrac{\partial \beta}{\partial \mu} \end{vmatrix} .$$

According to the coordinate transformation defined in equation (13), we find

$$\left| \frac{\partial(\alpha, \beta)}{\partial(\eta, \mu)} \right| = \frac{2\sqrt{(\alpha^2-1)(1-\beta^2)}}{a(\alpha^2-\beta^2)} \begin{vmatrix} \alpha & -\beta \\[1ex] \beta\sqrt{\dfrac{\alpha^2-1}{1-\beta^2}} & \alpha\sqrt{\dfrac{1-\beta^2}{\alpha^2-1}} \end{vmatrix} . \tag{A2}$$

The laws of transformation for the contravariant vector $\left(\dfrac{d\eta}{dt}, \dfrac{d\mu}{dt}\right)$ and the covariant vector $\left(\dfrac{\partial P}{\partial \eta}, \dfrac{\partial P}{\partial \mu}\right)$ are given as

$$\begin{vmatrix} \dfrac{d\alpha}{dt} \\ \dfrac{d\beta}{dt} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial(\alpha,\beta)}{\partial(\eta,\mu)} \end{vmatrix}^{T} \begin{vmatrix} \dfrac{d\eta}{dt} \\ \dfrac{d\mu}{dt} \end{vmatrix} , \tag{A3}$$

and

$$\begin{vmatrix} \dfrac{\partial\rho}{\partial\eta} \\ \dfrac{\partial\rho}{\partial\mu} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial(\alpha,\beta)}{\partial(\eta,\mu)} \end{vmatrix} \begin{vmatrix} \dfrac{\partial\rho}{\partial\alpha} \\ \dfrac{\partial\rho}{\partial\beta} \end{vmatrix} . \tag{A4}$$

where the superscript T in equation (A3) denotes the transposed matrix, and

$$\frac{d\eta}{dt} = u \ , \quad \frac{d\mu}{dt} = w \ , \quad \text{and} \quad \frac{a}{2}\sqrt{\frac{\alpha^2-\beta^2}{\alpha^2-1}}\,\frac{d\alpha}{dt} = V_\alpha \ . \tag{A5}$$

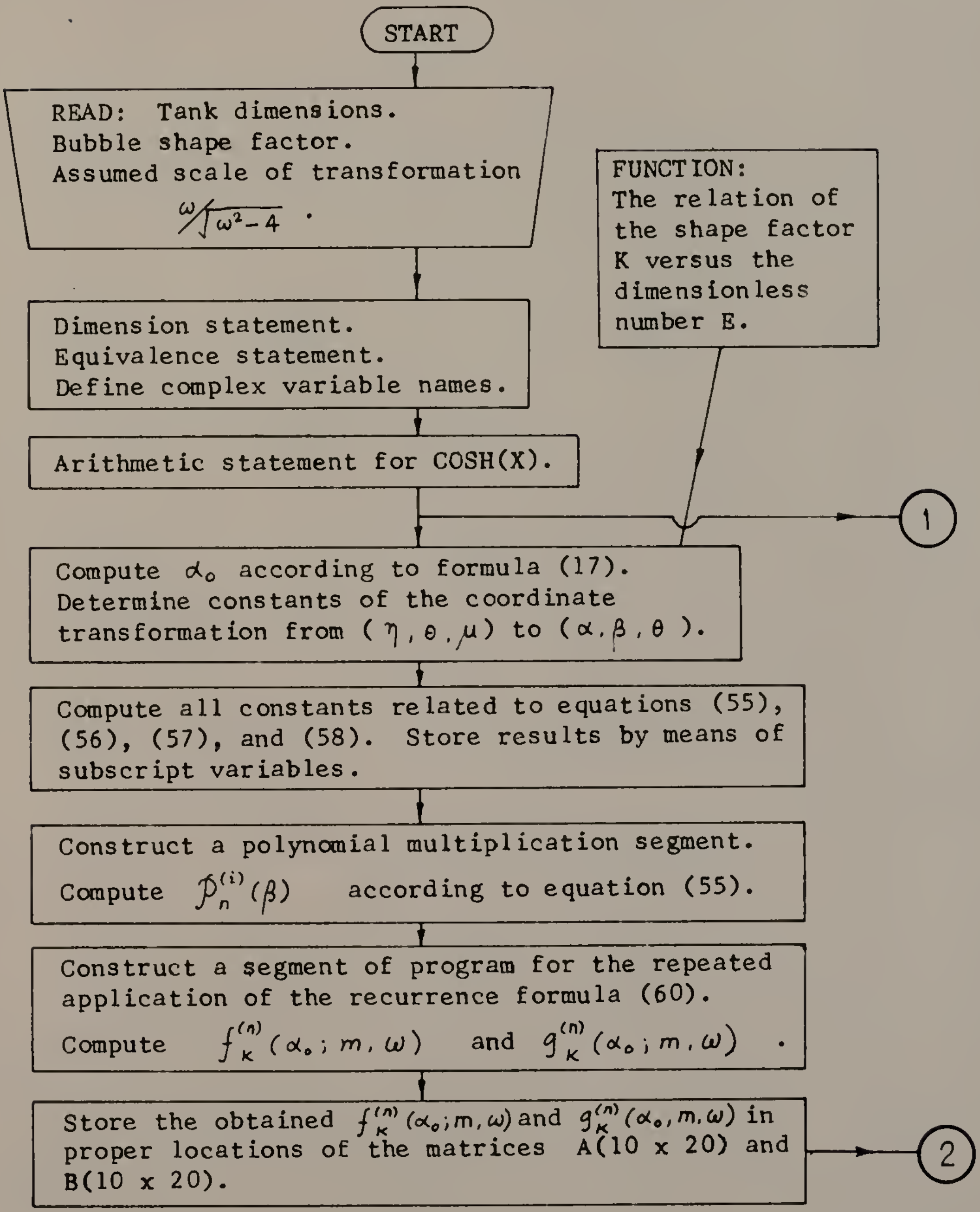
Hence, by using equations (A1), (A2), (A3), and (A4), we obtain

$$\frac{d\alpha}{dt} = \frac{1}{\omega^2-4}\left[\frac{4i\omega}{a^2}\,\frac{(\alpha^2-1)}{(\alpha^2-\beta^2)}\,\frac{\partial\rho}{\partial\alpha} + \frac{8\alpha}{a^2(\alpha^2-\beta^2)}\,\frac{\partial\rho}{\partial\theta} \right] . \tag{A6}$$

The expression for V_α follows immediately from equations (A5) and (A6).

APPENDIX B

FLOW CHART OF THE COMPUTING PROGRAM EMPLOYED IN CHAPTER III



READ: Tank dimensions.
Bubble shape factor.
Assumed scale of transformation
$\omega / \sqrt{\omega^2 - 4}$.

FUNCTION:
The relation of the shape factor K versus the dimensionless number E.

Dimension statement.
Equivalence statement.
Define complex variable names.

Arithmetic statement for COSH(X).

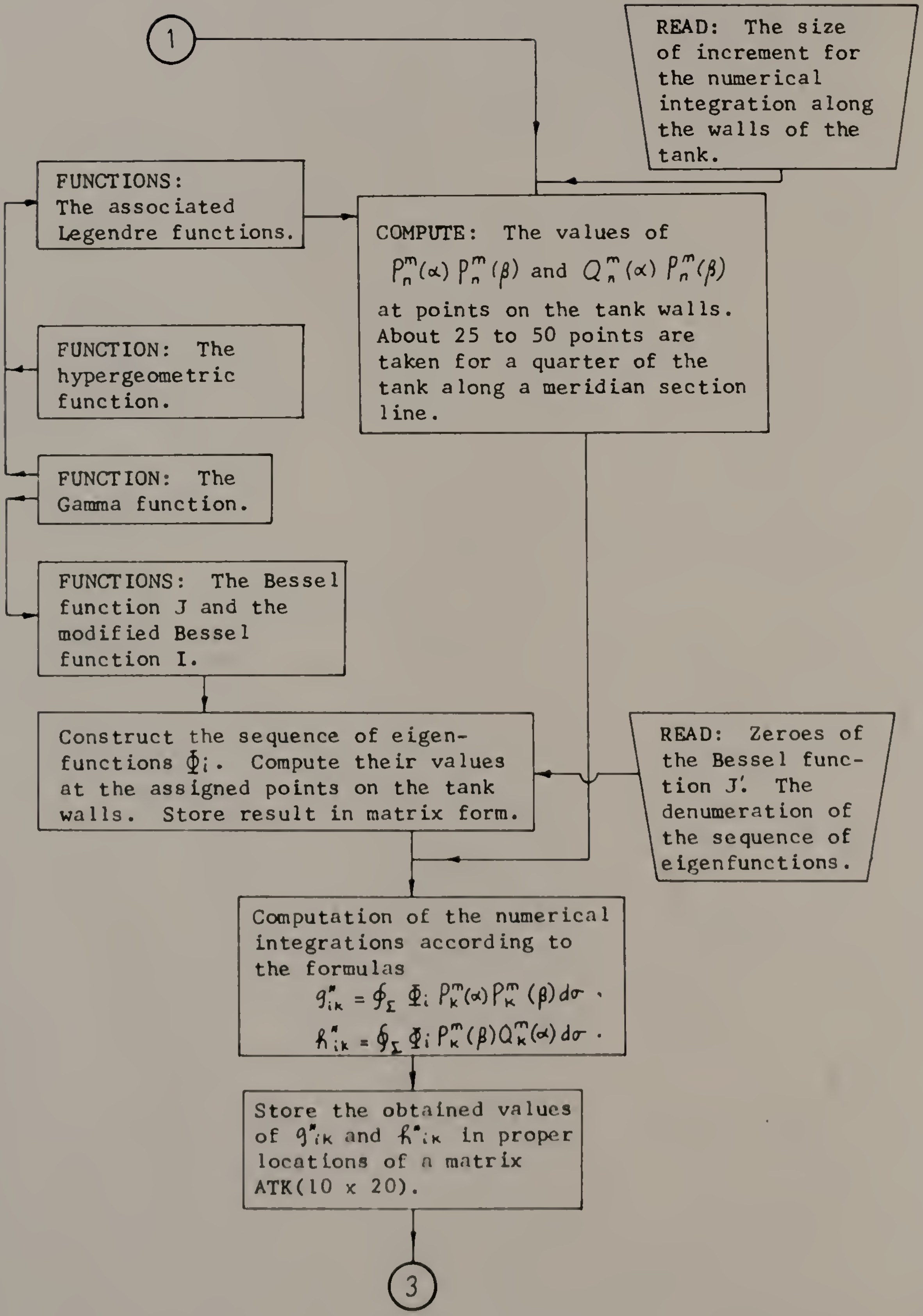
Compute α_0 according to formula (17).
Determine constants of the coordinate transformation from (η, θ, μ) to (α, β, θ).

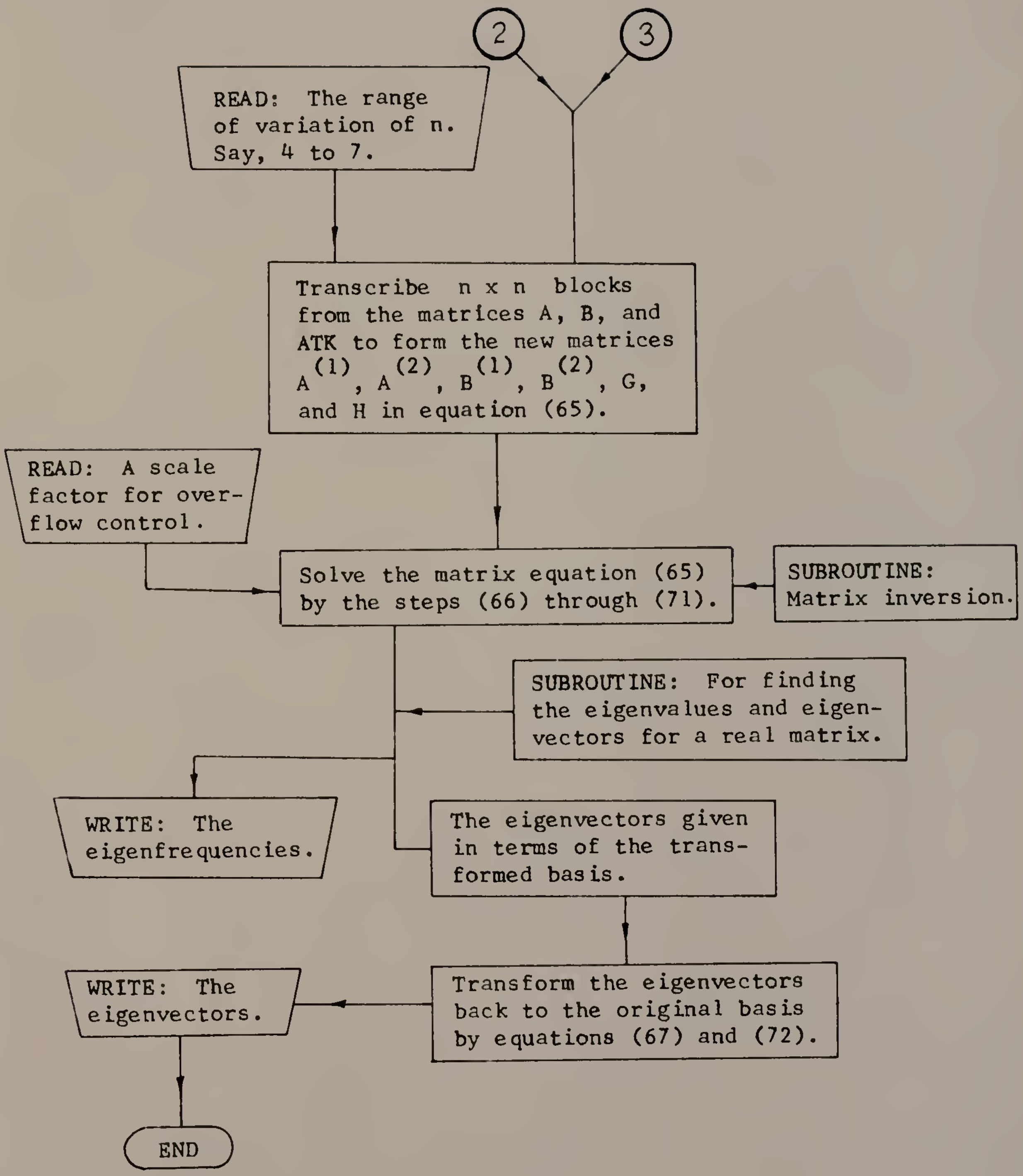
Compute all constants related to equations (55), (56), (57), and (58). Store results by means of subscript variables.

Construct a polynomial multiplication segment.
Compute $p_n^{(i)}(\beta)$ according to equation (55).

Construct a segment of program for the repeated application of the recurrence formula (60).
Compute $f_K^{(n)}(\alpha_0; m, \omega)$ and $g_K^{(n)}(\alpha_0; m, \omega)$.

Store the obtained $f_K^{(n)}(\alpha_0; m, \omega)$ and $g_K^{(n)}(\alpha_0, m, \omega)$ in proper locations of the matrices A(10 x 20) and B(10 x 20).


1
READ: The size of increment for the numerical integration along the walls of the tank.
FUNCTIONS: The associated Legendre functions.
COMPUTE: The values of $P_n^m(\alpha) P_n^m(\beta)$ and $Q_n^m(\alpha) P_n^m(\beta)$ at points on the tank walls. About 25 to 50 points are taken for a quarter of the tank along a meridian section line.
FUNCTION: The hypergeometric function.
FUNCTION: The Gamma function.
FUNCTIONS: The Bessel function J and the modified Bessel function I.
Construct the sequence of eigenfunctions Φ_i. Compute their values at the assigned points on the tank walls. Store result in matrix form.
READ: Zeroes of the Bessel function J'. The denumeration of the sequence of eigenfunctions.
Computation of the numerical integrations according to the formulas
$g_{ik}^m = \oint_\Sigma \Phi_i P_k^m(\alpha) P_k^m(\beta)\, d\sigma$.
$h_{ik}^m = \oint_\Sigma \Phi_i P_k^m(\beta) Q_k^m(\alpha)\, d\sigma$.
Store the obtained values of g_{ik}^m and h_{ik}^m in proper locations of a matrix ATK(10 x 20).
3


2
3
READ: The range of variation of n. Say, 4 to 7.
Transcribe n x n blocks from the matrices A, B, and ATK to form the new matrices $A^{(1)}$, $A^{(2)}$, $B^{(1)}$, $B^{(2)}$, G, and H in equation (65).
READ: A scale factor for overflow control.
Solve the matrix equation (65) by the steps (66) through (71).
SUBROUTINE: Matrix inversion.
SUBROUTINE: For finding the eigenvalues and eigenvectors for a real matrix.
WRITE: The eigenfrequencies.
The eigenvectors given in terms of the transformed basis.
WRITE: The eigenvectors.
Transform the eigenvectors back to the original basis by equations (67) and (72).
END

REFERENCES

[1] Rosenthal, D. K. The shape and stability of a bubble at the axis of a rotating liquid. J. Fluid Mech. 12 (1962), 358.

[2] Habip, L. M. On the mechanics of liquids in subgravity. Astronautica Acta 11 (1965), 401.

[3] Morgan, G. W. A study of motions in a rotating liquid. Proc. Roy. Soc. A, 206 (1951), 108.

[4] Phillips, O. M. Centrifugal waves. J. Fluid Mech. 7 (1960), 340.

[5] Struik, D. J. Lectures on Classical Differential Geometry. (1950), Addison-Wesley Press, Cambridge, Mass.

[6] Oser, H. Erzwungene Schwingungen in rotierenden Flüssigkeiten. Arch. Nat. Mech. Anal. 1 (1957), 81.

[7] Garabedian, P. R. Partial Differential Equations. (1964), John Wiley & Sons, New York.

[8] Benjamin, T. B., and Barnard, B. J. S. A study of the motion of a cavity in a rotating liquid. J. Fluid Mech. 19 (1964), 193.

BIOGRAPHICAL SKETCH

Sui-kwong Paul Pao was born on October 23, 1940, in Canton, China. He received the degree of Bachelor of Science in Engineering from the National Taiwan University in July, 1961. He enrolled in the Graduate School of the University of Florida in February, 1962. He worked as a graduate assistant in the Department of Engineering Mechanics until April, 1963, when he received the degree of Master of Science in Engineering. He worked as a research assistant in the same department in the following summer, and as a structural engineer in an engineering firm in New York, N.Y., between September, 1963, and December, 1964. He resumed his graduate study in the Department of Engineering Science and Mechanics of the University of Florida in January, 1965, and was granted the College of Engineering Fellowship.

Sui-kwong Paul Pao is married to the former Juliet Yung-li Zue. He is a member of Phi Kappa Phi.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, was was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April, 1967

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee:

Chairman